Faculty of Education
Department of Mathematics

Numerical Spectral Solutions for Some Differential Equations Via Special Kinds of Polynomials

Presented by

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Abstract

The thesis in its entirety consists of three chapters. In the first chapter, an introduction to orthogonal functions, focusing on Gegenbauer polynomials. In addition, it provides a brief overview of spectral methods, namely, tau, collocation, and galerkin. This required the presentation of definitions, concepts, relationships and theories that were employed to serve the needs of the calculations done in the thesis. The second chapter presents a new type of orthogonal polynomials called modified shifted Gegenbauer polynomials and their use in constructing operational matrices of derivatives. They have been used to find spectral solutions to some boundary value problems of second-order. The effectiveness of the method was achieved in finding analytical solutions for some linear and non-linear equations and the Bratu type equation. Finally, the third chapter employs the shift Gegenbauer-Galerkin method to solve the telegraph equations. We needed to deduce relations for shifted Gegenbauer polynomials and build new basis functions and evaluate matrices elements. Also, it discusses the study of convergence and error of the method.

Summary

The main **objectives** of this thesis can be summarized in the following points:

- A survey study on orthogonal polynomials in general and on Jacobi and Gegenbauer polynomials in particular.
- Collecting some important formulas concerned with the Gegenbauer polynomials.
- A comprehensive study on spectral methods, and their celebrated methods, namely, tau, collocation and Galerkin methods.
- Establishing operational matrices of derivatives of some basis functions.
- Implementing spectral algorithms for handling second-order boundaryvalue problems.
- Implementing a numerical spectral algorithm for treating the hyperbolic- telegraph type equation.
- Comparing our proposed algorithms with some other ones in the literaturedevoted to solving the same problems aiming to demonstrate their accuracy and applicability.

The thesis consists of three chapters and is organized as follows:

Chapter 1

The purpose of this chapter is to introduce some properties of orthogonal polynomials in general and of Gegenbauer polynomials in particular. Furthermore, an overview of spectral methods and their advantages over the other standard methods is presented.

Chapter 2

The main objectives of this chapter can be summarized in the following points:

- Establishing a new type of orthogonal polynomials, namely, modified shifted Gegenbauer polynomials. The established orthogonal polynomials generalize some other orthogonal polynomials that exist in the literature.
- Constructing the operational matrices of the modified shifted Gegenbauer polynomials.
- Employing the introduced operational matrices to solve the second-order BVPs.

The results of this chapter are published in:

H.T. Taghian, W.M. Abd-Elhameed, and Y.H. Youssri, A modified shifted Gegenbauer polynomials for the numerical treatment of second-order BVPs, Mathematical Sciences Letters **11**(1), (2022), pp. 1-12.

Chapter 3

The main objectives of this chapter can be listed as follows:

- Introducing a new combination of basis functions in terms of Gegenbauer polynomials that will be used as basis functions.
- Implementing and analyzing a numerical algorithm based on employing suitable combinations of Gegenbauer polynomials to solve the hyperbolic telegraph type equation.
- Discussion of the convergence and error analysis of the proposed Gegenbauer expansion.

The results of this chapter are published in:

H.T. Taghian, W.M. Abd-Elhameed, G.M. Moatimid, Y.H. Youssri, Shifted Gegenbauer–Galerkin algorithm for hyperbolic telegraph type equation, International Journal of Modern Physics C, **32**(9), pp. 1-20

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Fundamentals

Chapter 1

Fundamentals

Numerical methods for differential equations can be classified into the *local* and *global* categories. The finite-difference and finite-element methods are based on local arguments, whereas the spectral method is global in character. In practice, finite-element methods are particularly well suited to problems in complex geometries, whereas spectral methods can provide superior accuracy, at the expense of domain flexibility. There are many numerical approaches, such as finite elements and spectral elements, which combine advantages of both the global and local methods (see, [1]). However, in this thesis, we shall restrict our attention to the global spectral methods.

Spectral methods, in the context of numerical schemes for differential equations belong to the family of weighted residual methods (WRMs), which are traditionally regarded as the foundation of many numerical methods such as finite element, spectral, finite volume, boundary element (see, for example, [2,3]). WRMs represent a particular group of approximation techniques, in which the residuals (or errors) are minimized in a certain way and thereby leading to specific methods including Galerkin, Petrov-Galerkin, collocation and tau formulations.

The aim of spectral methods is to approximate functions (solutions of differential equations) by means of truncated series of orthogonal polynomials. There are three well-known methods of spectral methods, namely, tau, collocation and Galerkin methods (see, for example, [4, 5]). The choice of the suitable used spectral method suggested for solving the given equation depends