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Mathematics Department

On Oscillatory and Asymptotic Behavior of Solutions of Some Dynamic Equations on Time Scales.

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(Pure Mathematics)

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Contents

Contents i

Summary iii

1 Preliminaries 1
1.1 Initial Value Problems 1
1.2 Oscillation and Nonoscillation 5
1.3 Introduction to Neutral-Differential Equations 6
1.4 Some Oscillatory and Nonoscillatory Phenomena Caused by Deviating Arguments 7
1.5 Comparison Theorems and Oscillation 8
1.6 Oscillatory and asymptotic behavior of solutions of Second Order Neutral Differential Equations with ”maxima” 9
1.7 oscillation of second-order differential equations with a sublinear or a superlinear neutral terms 12

2 Calculus on time scales and some previous results 18
2.1 Calculus on Time Scales 19
2.1.1 Basic Definitions 19
2.1.2 Differentiation 21
2.1.3 Examples and Applications 25
2.1.4 Integration 28
2.1.5 Chain Rules 33
2.1.6 The Exponential Function 35
2.1.7 Initial value problems 38
2.2 Oscillation of integro-dynamic equations on time scales 41

3 Oscillation results for second-order neutral integro-dynamic equations with damping 45

3.1 New oscillation results for second order integro- dynamic equations with damping and distributed deviating arguments 46

3.1.1 Basic Concepts and Lemmas 47

3.1.2 Main results 51

3.1.3 Example 60

3.2 Oscillation results for second-order mixed neutral integro-dynamic equations with damping and a nonpositive neutral term on time scales 61

3.2.1 Several lemmas 62

3.2.2 Main results 66

3.2.3 Example 76

4 Oscillatory Behavior of second-order neutral dynamic equations with maxima and mixed arguments. 78

4.1 Basic Concepts and Lemmas 78

4.2 Basic lemmas 79

4.3 Main results 82

4.4 Examples 96

5 Oscillatory and asymptotic behavior of solutions for second-order mixed nonlinear integro-dynamic equations with maxima 99

5.1 Basic concepts and Lemmas 100

5.2 Main results 101

5.3 Example 122

6 Oscillatory behavior of second-order integro-dynamic equations with maxima and superlinear or sublinear neutral terms 125

6.1 Basic concepts and lemmas 125

6.2 Main results 130

6.3 Example 142

Bibliography 145

Abstract

Studing the dynamic equations on time scales was introduced by Stefan Hilger [3٧]. It is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is a notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics. Further, studying time scales lead to several important applications, e.g., insect population models, neural networks, and heat transfer. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers. When the time scale equals the set of real numbers, the obtained results yield results of ordinary differential equations, while when the time scale equals the set of integers, the obtained results yield results of difference equations. The new theory of the so - called "dynamic equation" is not only unify the theories of differential and difference equations, but also extends the classical cases to the so - called q - difference equations (when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0,\ q > 1\}$ or $\mathbb{T} = q^{\overline{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$) which have important applications in quantum theory (see [3٨]).

A neutral differential equation with deviating arguments is a differential equation in which the highest order derivative of the unknown function appears with and without deviating arguments. In the last two decades,

This thesis is devoted to

1. Illustrate the new theory of Stefan Hilger by giving an introduction to the theory of dynamic equations on time scales.
2. Summarize some of the recent developments in oscillation of second order neutral differential equations with "maxima", oscillation of second-order differential equations with a sublinear or a superlinear neutral terms, oscillation of second order nonlinear integro- dynamic equations on time scales.
3. Establish new sufficient conditions to study the oscillatory and the asymptotic behavior of solutions of second order nonlinear neutral dynamic equations with maxima and second order nonlinear mixed neutral integro-dynamic equations with maxima on time scales, so that the obtained results are more generalized than those obtained in previous studies.
4. Give some examples to illustrate the relevance of our results.

This thesis consists of six chapters :-

Chapter 1, is an introductory chapter that contains the basic concepts of theory of functional differential equations, some previous results of the oscillatory and the asymptotic behavior of solutions of second order neutral differential equations with "maxima" and oscillation of second-order differential equations with a sublinear or a superlinear neutral terms.

In Chapter 2, we give an introduction to the theory of dynamic equations on time scales, differentiation and integration, and some examples of time scales. Moreover, we present various properties of generalized exponential function on time scales. Additionally, some previous studies for the oscillatory and asymptotic behavior of second order integro- dynamic equations on time scales are presented.

Chapter 3, consists of two sections. In the first section, we present some new oscillation criteria for the second order neutral integro-dynamic equation with damping and distributed deviating arguments:

In Chapter 4, Some new oscillation criteria for the second order neutral dynamic equation with maxima and mixed arguments

Chapter 5, Concerned with the oscillatory and asymptotic behavior for solutions of the second-order mixed nonlinear integro-dynamic equations with "maxima"

Chapter 6, deals with the oscillatory and the asymptotic behavior of the solutions of the second-order neutral nonlinear integro-dynamic equations with maxima and superlinear or sublinear neutral terms on time scales

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Hebat-Allah Mohammed Arafa

Summary

Studying the dynamic equations on time scales was introduced by Stefan Hilger [37]. It is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is a notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics. Further, studying time scales lead to several important applications, e.g., insect population models, neural networks, and heat transfer. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers. When the time scale equals the set of real numbers, the obtained results yield results of ordinary differential equations, while when the time scale equals the set of integers, the obtained results yield results of difference equations. The new theory of the so - called "dynamic equation" is not only unify the theories of differential and difference equations, but also extends the classical cases to the so - called q - difference equations (when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0, q > 1\}$ or $\mathbb{T} = q^{\overline{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$) which have important applications in quantum theory (see [38]).

A neutral differential equation with deviating arguments is a differential equation in which the highest order derivative of the unknown function appears with and without deviating arguments. In the last two decades, there has been increasing interest in obtaining sufficient conditions for oscillation (nonoscillation) of the solutions of dynamic equations on time scales. So we choose the title of the thesis "On Oscillatory and Asymptotic Behavior of Solutions of Some Dynamic Equations on Time Scales". We aim to establish new criteria for the oscillatory and the asymptotic behavior of solutions of second order nonlinear neutral integro-dynamic equations with "maxima" on a time scale $\mathbb{T}$.

This thesis is devoted to

1. Illustrate the new theory of Stefan Hilger by giving an introduction to the theory of dynamic equations on time scales.

2. Summarize some of the recent developments in oscillation of second order neutral differential equations with "maxima", oscillation of second-order differential equations with a sublinear or a superlinear neutral terms, oscillation of second order nonlinear integro- dynamic equations on time scales.

3. Establish new sufficient conditions to study the oscillatory and the asymptotic behavior of solutions of second order nonlinear neutral dynamic equations with maxima and second order nonlinear mixed neutral integro-dynamic equations with maxima on time scales, so that the obtained results are more generalized than those obtained in previous studies.

4. Give some examples to illustrate the relevance of our results.

This thesis consists of six chapters :-

Chapter 1, is an introductory chapter that contains the basic concepts of theory of functional differential equations, some previous results of the oscillatory and the asymptotic behavior of solutions of second order neutral differential equations with "maxima" and oscillation of second-order differential equations with a sublinear or a superlinear neutral terms.

In Chapter 2, we give an introduction to the theory of dynamic equations on time scales, differentiation and integration, and some examples of time scales. Moreover, we present various properties of generalized exponential function on time scales. Additionally, some previous studies for the oscillatory and asymptotic behavior of second order integro- dynamic equations on time scales are presented.

Chapter 3, consists of two sections. In the first section, we present some new oscillation criteria for the second order neutral integro-dynamic equation with damping and distributed deviating arguments:

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + p(t)(z^{\Delta}(t))^{\gamma} + \int_{t_0}^{t} G(t, s, x(\eta(t, s)))\Delta s = 0,$$

where

$$z(t) = x(t) + \int_{t}^{\tau_1^{-1}(t)} F_1(s, x(\tau_1(s)))\Delta s + \int_{t}^{\tau_3(t)} F_2(s, x(\tau_2(s)))\Delta s.$$

on a time scale $\mathbb{T}$. In the second section, we establish some new oscillation criteria for the second order mixed neutral integro dynamic equation with damping and a non positive neutral term of the form:

$$(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} + p(t)(v^{\Delta}(t))^{\gamma} + g(t, x(\tau(t))) + \int_{0}^{t} a(t,s) f(s, x(s))\Delta s = 0,$$

where

$$v(t) = x(t) - p_1(t)x(\eta_1(t)) + p_2(t)x(\eta_2(t)),$$

on a time scale $\mathbb{T}$. Oscillation behavior of these equations is not studied before. Our results not only improve and extend some results established by [4], [6], [7], [26] ,[28], [29] and [47], but also can be applied to some oscillation problems that are not studied before.
The results of this chapter are presented in:

- The 11^{th}symposium of the fractional calculus and applications group "one day conference in mathematics", July (20) (2019), Faculty of science, Alexandria university.
- 3^{rd} international conference of mathematics, November 26-27 (2020).

Moreover, some results of this chapter accepted for publication in:
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In Chapter 4, Some new oscillation criteria for the second order neutral dynamic equation with maxima and mixed arguments

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + q(t) \max_{s\in[\xi(t),\eta(t)]_{\mathbb{T}}} x^{\gamma}(s) = 0,$$

where

$$z(t) = x(t) + p_1(t)x(\tau_1(t)) + p_2(t)x(\tau_2(t)), \qquad t \in [t_0, \infty)_{\mathbb{T}}.$$

on a time scale $\mathbb{T}$ are established. New comparison theorems are obtained that allow us to examine the oscillation of the given second order dynamic equation by examining the oscillatory behavior of first order dynamic ones. The obtained results extend and complement some known ones in literature. Examples are given to illustrate the importance of our results.
The results of this chapter are published in:
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DOI: 10.1080/09720502.2020.1756047 [9].

Chapter 5, Concerned with the oscillatory and asymptotic behavior for solutions of the second-order mixed nonlinear integro-dynamic equations with "maxima"

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + \int_{0}^{t} a(t,s)f(s,x(s))\Delta s + \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s) = 0,$$

where

$$z(t) = x(t) + p_1(t)x(\eta_1(t)) + p_2(t)x(\eta_2(t)),$$

on a time scale $\mathbb{T}$, The oscillatory behavior of this equation hasn't been discussed before. The obtained results not only present some new criteria for such kind of neutral differential equations and neutral difference equations as special cases but also extend some results obtained by Grace et al. [4] and [29].
Results of this chapter are published in:
Filomat, 33 (10) (2019), 2907-2929. [8]

Chapter 6, deals with the oscillatory and the asymptotic behavior of the solutions of the second-order neutral nonlinear integro-dynamic equations with maxima and superlinear or sublinear neutral terms on time scales

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + \int_{0}^{t} a(t,s)f(s,x(s))\Delta s + \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s) = 0,$$

where

$$z(t) = x(t) + p_1(t)x^{\lambda_1}(\eta_1(t)) + p_2(t)x^{\lambda_2}(\eta_2(t)), \qquad t \in [0,+\infty)_{\mathbb{T}}.$$

on a time scale $\mathbb{T}$. The obtained results are new for both the discrete and continuous cases. Furthermore, our results extend known ones in the literature. These results accepted for publication in:
Sarajevo Journal of Mathematics, Accepted [10].

Chapter 1

Preliminaries

This chapter is considered as a background for the subject of this thesis. We give a survey of the related material included in previous studies and present some basic concepts for the theory of functional differential equations. Also, we sketch some preliminary results that can be used in this thesis and introduce some of the recent developments in oscillation theory of second order neutral differential equations.

1.1 Initial Value Problems

In this section, we give the definitions of ordinary and functional differential equations.

Definition 1.1.1 *An ordinary differential equation (ODE) is an equation that contains a function of only one independent variable, and some of its derivatives with respect to that variable.*

Definition 1.1.2 *[45] A functional equation (FE) is an equation involves an unknown function for different argument values. The difference between the argument values of the unknown function and t in the FE are called argument deviations. If all argument deviations are constants, then the FE is called a difference equation.*

Example 1.1.1 *The equations $x(3t)+4t^3x(6t) = 4$, and $x(t) = e^t x(t+1)-[x(t-3)]^2$ are examples of FEs.*

Combining Definition 1.1.1 and Definition 1.1.2, we obtain the following definition of functional differential equation (FDE), or equivalently, differential equation with deviating arguments.

Definition 1.1.3 *[45] A functional differential equation is an equation contains an unknown function and some of its derivatives for different argument values. The order of a FDE is the order of the highest derivative of the unknown function. So, a FE is a functional differential equation of order zero.*

Definition 1.1.4 *The ordinary differential equation*

$$x'(t) = f(t, x(t)) \tag{1.1}$$

together with the condition

$$x(t_0) = x_0, \tag{1.2}$$

is called an initial value problem. (1.2) is called an initial condition, t_0 is an initial point.

It is well known that under certain assumptions on f, the initial value problem (1.1) and (1.2) has the unique solution

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))ds \qquad for \qquad t \geq t_0 \tag{1.3}$$

Definition 1.1.5 *The differential equation of the form*

$$x'(t) = f(t, x(t), x(t-\tau)) \qquad with \qquad \tau > 0 \qquad and \qquad t \geq t_0, \tag{1.4}$$

in which the right-hand side depends on the instantaneous position $x(t)$ and the position at τ units back $x(t-\tau)$, is called an ordinary differential equation with delay or a delay differential equation.

Whenever necessary, we consider the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, \tag{1.5}$$

equivalent to (1.4). In order to find a solution of (1.4), we need to have a known function φ on $[t_0-\tau, t_0]$, instead of the initial condition $x(t_0) = x_0$. The basic initial value problem for a delay differential equation on the interval $[t_0, T]$, $T < \infty$, is defined by (1.4) and the initial condition

$$x(t) = \varphi(t) \quad \text{for all} \quad t \in E_{t_0}, \tag{1.6}$$

where t_0 is an initial point and $E_{t_0} = [t_0 - \tau, t_0]$, and the function φ on E_{t_0} is called the initial function. Usually, it is assumed that $\varphi(t_0 + 0) = \varphi(t_0)$. By a one-sided derivative, we mean the derivative at that side of the interval. Under general assumptions, the existence and uniqueness of the solution of the initial value problem (1.4) and (1.6) can be established (see for example Ladas[34]). The solution is sometimes denoted by $x(t, \varphi)$. In case of variable delay $\tau = \tau(t) > 0$, the initial set E_{t_0} has the form:

$$E_{t_0} = \{t_0\} \cup \{t - \tau(t) : t - \tau(t) < t_0, t \geq t_0\}.$$

To determine the solution on the interval $[t_0, T]$, we use the initial set

$$E_{t_0T} = \{t_0\} \cup \{t - \tau(t) : t - \tau(t) < t_0, t_0 \leq t \leq T\}.$$

Example 1.1.2 *[2] for the equation*

$$x'(t) = f(t, x(t), x(t - cos^2 t)),$$

$t_0 = 0$, $E_0 = [-1, 0]$, *the initial function* φ *must be given on* $[-1, 0]$.

The dependence of E_{t_0} on t_0, can be seen in the following example.

Example 1.1.3 *[2] for the equation*

$$x'(t) = a\, x(t/2),$$

we have $\tau(t) = t/2$ *so that*

$$E_0 = \{0\} \text{ and } E_1 = [1/2, 1].$$

Now, consider the differential equation of order n with l deviating arguments, $x^{(m_0)}(t) = f(t, x(t), ..., x^{(m_0-1)}(t), x(t - \tau_1(t)), ..., x^{(m_1-1)}(t - \tau_1(t)), ...,$

$$x(t - \tau_l(t)), ..., x^{(m_l-1)}(t - \tau_l(t))), \tag{1.7}$$

where the deviations $\tau_i(t) > 0$, and $n = \max\{\max_{1\leq i\leq l}(m_i - 1), m_0\}$. In order to formulate the initial value problem for (1.7), each deviation $\tau_i(t)$ defines an initial set $E_{t_0}^{(i)}$ as:

$$E_{t_0}^{(i)} = \{t_0\} \cup \{t - \tau_i(t) : t - \tau_i(t) < t_0, t \geq t_0\},$$

where $E_{t_0} = \cup_{i=1}^{l} E_{t_0}^{(i)}$. On E_{t_0}, continuous functions φ_k, $k = 0, 1, ..., \lambda$, must be given with $\lambda = \max_{1 \leq i \leq l}(m_i - 1)$. In applications, we often (not generally) consider the initial conditions,

$$\varphi_k(t) = \varphi_0^{(k)}(t) \quad \text{for} \quad t \in E_{t_0} \quad k = 0, 1, ..., \lambda.$$

For the nth order differential equation, initial values $x_0^{(k)}$, $k = 0, 1, 2, ..., n-1$ must be given. Now let $x_0^{(k)} = \varphi_k(t_0)$, $k = 0, 1, 2, ..., \lambda$. If $\lambda < n - 1$, then the numbers $x_0^{(\lambda+1)}, ..., x_0^{(n-1)}$ are given. If the point t_0 is an isolated point of E_{t_0}, then $x_0^{(0)}, ..., x_0^{(n)}$ are given. For (1.7), the basic initial value problem consists of the determination of an $(n-1)$ times differentiable function x that satisfies (1.7) for $t > t_0$,

$$x^{(k)}(t_0 + 0) = x_0^{(k)}, \quad k = 0, 1, 2, ..., n-1$$

and

$$x^{(k)}(t - \tau_i(t)) = \varphi_k(t - \tau_i(t)) \quad \text{for} \quad t - \tau_i(t) < t_0,$$

where $k = 0, 1, 2, ..., \lambda$ and $i = 0, 1, 2, ..., l$. At the point $t_0 + (k-1)\tau$, the derivatives $x^{(k)}(t)$ are in general discontinuous, but the derivatives of lower order are continuous.

Example 1.1.4 *[2] consider the equation*

$$x''(t) = f(t, x(t), x'(t), x(t - cos^2(t)), x(\frac{t}{2})). \tag{1.8}$$

In this case, we have $n = 2$, $l = 2$ and $\lambda = 0$. For $t_0 = 0$, the initial sets are $E_0^{(1)} = [-1, 0]$, and $E_0^{(2)} = \{0\}$. Hence $E_0 = [-1, 0]$. On E_0, the initial function φ_0 is given. Also the initial values $x_0^{(0)} = \varphi_0(0)$, and $x_0^{(1)} = \varphi_1(0)$ are given numbers.

A classification method for (1.7) was proposed by Kamenskiǐ [39]. Let $\beta = m_0 - \lambda$. If $\beta > 0$, (1.7) is called an equation with delay (retarded arguments), if $\beta = 0$, it is called an equation of neutral type, and if $\beta < 0$, it is called an equation of advanced type.

Example 1.1.5 *[2] The equations*

$$x'(t) + a(t)x(t - \tau) = 0,$$