On Some Problems in the Fractional Calculus

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To My Mother and

Prof. Ahmed M. A. El-Sayed

with love and gratitude

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Contents

Introduction

Introduction

In this thesis, a study will be made about the use of fractional calculus in solving several problems of mathematical and physical importance involving ordinary and partial differential equations of fractional-order. These equations are associated with distinct types of physical phenomena such as: relaxation process, oscillation process, diffusion process and wave propagation etc. Consequently, they are of fundamental importance in many branches of physics. They are also of considerable significance from a mathematical point of view. In the last few years the applications of fractional calculus has continued to develop rapidly. Nowadays, there exists a great number of articles entirely devoted to the applications of fractional calculus. Some of the main directions are: (i) Differential and integral equations (ii) Partial differential equations (iii) Special functions, and other branches of analysis.

The *first Chapter* consists of a sequence of definitions and preliminary properties interleavened with many examples which will illustrate the main ideas. We begin by reviewing the most basic and known definitions of fractional integral (Riemman-Liouville and finite Weyl operators) and its properties, and compare between two definitions of fractional derivatives (Riemman-Liouville and El-Sayed approaches) from the viewpoint of formulation and the use in applied problems. Also the negative-direction fractional calculus will be considered and some of its properties. After introducing definitions and preliminary properties we briefly cover the background material which is required for the sequel.

In *Chapter 2* , we develop a theory of ordinary differential equations with memory. The nonhomogeneous differential equation with memory function $k(t)$ is given by

$$Dx(t) = -\int_a^t k(t-s)\, x(s)\, ds \ + \ f(t) \ , \ t > a.$$

This equation can be generalized to be

$$D_a^\alpha x(t) = -\lambda \int_a^t k(t-s)\, x(s)\, ds \ + \ f(t) \ , \quad t > a, \ \ \alpha \in (0,1],$$

where D_a^α is the fractional order derivative.

Let $\alpha \in (0,1]$ and $J = [a,b]$, $0 \le a < b < \infty$. Based on the definition of fractional order derivative D_a^α, the finite Weyl fractional derivative W_b^α and the formulation of the initial value problems of differential equations of arbitrary (fractional) orders we study the following nonhomogeneous initial value problems of fractional order:

(1) Forward problem

$$(II_1) \quad \begin{cases} D_a^\alpha\, x(t) &= -\lambda \int_a^t k(t,s)\, x(s)\, ds \ + \ f(t) \quad , \quad t > a \\ x(a) &= x_0 \end{cases}$$

(2) Forward-Backward problem

$$(II_2) \quad \begin{cases} D_a^\alpha\, x(t) &= -\lambda \int_t^b k(s,t)\, x(s)\, ds \ + \ f(t) \quad , \quad t \in (a,b) \\ x(a) &= x_0 \end{cases}$$

(3) Backward-Forward problem

$$(II_3) \quad \begin{cases} W_b^\alpha\, x(t) &= -\lambda \int_a^t k(t,s)\, x(s)\, ds \ + \ f(t) \quad , \quad t < b \\ x(b) &= x_b, \end{cases}$$

in the space $C(J,X)$, the space of continuous functions defined on J with values in the Banach space X. The existence of a unique solution of each of the previous problems in $C(J,X)$ will be proved and some properties of this solutions will be studied. As a special case of (II_1) we consider the following two cases of equation with memory

$$(II_1^*)\quad \begin{cases} D_a^\alpha\, x(t) &= -\,\lambda\, \int_a^t k(t-s)\, x(s)\, ds \qquad , \quad t > a \\ x(a) &= x_0, \end{cases}$$

$$(II_1^{**})\quad \begin{cases} D_a^\alpha\, x(t) &= -\,\lambda\, \int_a^t k(t-s)\, D^\beta\, x(s)\, ds \qquad , \quad 0 < \beta \leq \alpha \leq 1 \\ x(a) &= x_0. \end{cases}$$

with the memory function $k(t)$.

We end this chapter by showing how some of the results presented in the previous material can be used to find a unique solution to certain cases concerning special functions in the kernel. In brief we study the initial value problem

$$(II_1^{***})\quad \begin{cases} D^\alpha\, x(t) &= -\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\, {}_1F_1(\delta;\gamma;c(t-s)\, x(s)\, ds \;=\; {}_cS_\delta^\gamma\, x(t),\ \ \alpha,\gamma \in (0,1] \\ x(a) &= x_0, \end{cases}$$

where ${}_1F_1$ is the confluent hypergeometric function.

In the following two chapters we begin to demonstrate the power and usefulness of the results obtained in Chapter 2.

Chapter 3, contains the first application of Chapter 2. Here we formulate new general models describing some intermediate physical processes :

By suitable choice of the function $k(t)$, we formulate the two models

1. The fractional-order relaxation-oscillation model

$$(III_1)\quad \begin{cases} D^\alpha\, x(t) &= -\,\rho^{\alpha+\beta}\, I^\beta\, x(t) \;+\; I^\beta\, f(t) \qquad , \; t > 0 \\ x(0) &= x_0 \end{cases}$$

2. The forward-backward fractional-order relaxation-oscillation model

$$(III_2) \quad \begin{cases} D^{\alpha}\, x(t) &= -\, \rho^{\alpha+\beta}\, W_b^{-\beta}\, x(t) \qquad , \; t \in (0,b) \\ x(0) &= x(b), \end{cases}$$

that represent the relaxation-oscillation process. As special cases we study the following models

(i) The relaxation model;

(ii) The fractional-order relaxation model;

(iii) The oscillation model;

(iv) The fractional-order oscillation model;

(v) The backward relaxation model;

(vi) The fractional-order backward relaxation model;

(vii) The backward oscillation model.

In *Chapter 4*, we deal with another application of Chapter 2, initial and boundary value problems (IVPs and BVPs) of the Sturm-Liouville equations of fractional-order.

Let $p(t)$, $q(t)$ and $r(t)$ be continuous functions on the interval J such that $p(t) \in C^1(J, R)$ and $p(t) \neq 0$ for every $t \in J$ and λ is a parameter independent of t. Let $\beta, \alpha \in (0,1]$, $\beta \geq \alpha$.

The general Sturm-Liouville BVP can be defined by any one of the following models:

$$(IV_3) \quad \begin{cases} W_b^{\alpha}\, \left(p(t)\, \frac{dx(t)}{dt} \right) + W_b^{-(\beta-\alpha)}\, (\lambda\, q(t) + r(t))\, x(t) = 0 \qquad , \qquad t \in (a,b) \\ x(a) = x_0 \quad , \quad x'(b) = 0 \end{cases}$$

$$(IV_4) \quad \begin{cases} D_a^{\alpha}\, \left(p(t)\, \frac{dx(t)}{dt} \right) + I_a^{\beta-\alpha}\, (\lambda\, q(t) + r(t))\, x(t) = 0 \qquad , \qquad t \in (a,b) \\ x(b) = x_b \quad , \quad x'(a) = 0 \end{cases}$$

The general Sturm-Liouville IVP can be defined by the model:

$$(IV_6)\quad \begin{cases} D_a^{\alpha}\ (\ p(t)\ D_a^{\gamma}\ x(t)\ )\ +\ I_a^{\beta-\alpha}\ (\lambda\ q(t)\ +\ r(t))\ x(t) & =\ 0 \quad , \quad t \in (a,b),\ \gamma \in (0,1] \\ x(a)\ =\ x_0 \quad , \quad x'(a)\ =\ 0 & \end{cases}$$

Here we study the sufficient conditions for the existence, uniqueness and continuous dependence of the solution of each BVP and IVP stated above and then we find an explicit solution for them. Also the continuation to the BVPs and IVPs of Sturm-Liouville equation of integral order will proved.

In Chapters 5 to 6 we study some intermediate presses problems of physical importance. In *Chapter 5*, we study certain models describing some intermediate physical processes such as: Fractional-order Convection-Diffusion process

$$(V_1)\quad \begin{cases} \frac{\partial\ u(x,t)}{\partial\ t} & =\ {}_xW_b^{-\beta}\ \frac{\partial^2\ u(x,t)}{\partial\ x^2}, \quad \beta \in (0,1]\ ,\ t \in (0,T)\ ,\ x \in (0,b) \\ u(x,0) & =\ u_0(x) \\ u(b,t)\ (\text{or } u(0,t)\ ) & =\ u_x(b,t)\ =\ 0 \end{cases}$$

Fractional-order Transport-Diffusion process

$$(V_2)\quad \begin{cases} \frac{\partial\ u(x,t)}{\partial\ t} & =\ {}_xI^{\beta}\ \frac{\partial^2\ u(x,t)}{\partial\ x^2}, \quad \beta \in (0,1]\ ,\ t \in (0,T)\ ,\ x \in (0,b) \\ u(x,0) & =\ u_0(x) \\ u(0,t)\ (\text{or } u(b,t)\ ) & =\ u_x(0,t)\ =\ 0 \end{cases}$$

The method we used consists of two principal steps:

1. We prove that the operator ${}_xW_b^{-\beta}\ \frac{\partial^2\ u(x,t)}{\partial x^2}$ with the domain

$$\begin{cases} u(x,t); \quad u(x,t)\ \in\ H^2(0,b)\ \cap\ C^1((0,T);L_2(0,T)) \\ u(b,t)\ \ =\ u_x(b,t)\ =\ 0, \qquad \forall\ t \geq 0\ , \end{cases}$$

or the domain

$$\begin{cases} u(x,t); \quad u(x,t) \in H^2(0,b) \cap C^1((0,T);L_2(0,T)) \\ u(0,t) = u_x(b,t) = 0, \qquad \forall\, t \geq 0 \end{cases}$$

and the operator ${}_xI^\beta \ \frac{\partial^2\ u(x,t)}{\partial x^2}$ with the domain

$$\begin{cases} u(x,t); \quad u(x,t) \in H^2(0,b) \cap C^1((0,b);L_2(0,b)) \\ u(0,t) = u_x(0,t) = 0, \qquad \forall\, t \geq 0 \end{cases}$$

or the domain

$$\begin{cases} u(x,t); \quad u(x,t) \in H^2(0,b) \cap C^1((0,b);L_2(0,b)) \\ u(b,t) = u_x(0,t) = 0, \qquad \forall\, t \geq 0 \end{cases}$$

generate a semigroup of bounded linear operators.

2. We use the property (1) to prove the existence of a unique solution of each of the previous problems by applying the semigroup theory.

The continuation to the usual convection, diffusion and transport processes will be proved.

In *Chapter 6,* we give the more convenient definition of the abstract fractional-order diffusion-wave problems as

$$(D\text{-}W) \quad \begin{cases} D^\alpha\ u(t) = I^\beta\ A\ u(t) \qquad t > 0,\ \alpha \in (0,1],\ \beta > 0 \\ u(0) = u_0 \end{cases}$$

and

$$({}_-D\text{-}W) \quad \begin{cases} S^\alpha\ u(t) = S^{-\beta}\ A\ u(t) \qquad t < 0,\ \alpha \in (0,1],\ \beta > 0 \\ u(0) = u_0, \end{cases}$$

and then study the continuation properties of this problems to classical abstract diffusion, wave problems and fractional-order diffusion-wave problem, where S^α is the negative-direction fractional derivatives and A is a closed linear operator with domain $D(A)$ dense in the Banach space X. And as special cases, we study models that represents the following processes

(i) Abstract diffusion process;

(ii) Abstract fractional-order diffusion process;

(iii) Abstract wave process;

(iv) Abstract fractional-order wave process;

(v) The abstract backward diffusion process;

(vi) The abstract backward fractional-order diffusion process ;

(vii) The abstract backward wave process;

(viii) The abstract backward fractional-order wave process.

Chapter 1

Preliminaries