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Contents

Chapter 1

Preliminaries

The purpose of this chapter is to present a short survey of some needed definitions and theories of the material used in this thesis.

1.1 Some basic concepts of topological structures

The aim of this section is to collect the relevant definitions and results from topology about interior, closure, boundary, separation axioms and mappings.

Definition 1.1.1 *[35] Let X be a nonempty set. A class τ of subsets of X is called a topology on X if it satisfies the following axioms:*

1. *$X, \phi \in \tau$,*
2. *an arbitrary union of the members of τ is in τ,*
3. *the intersection of any two sets in τ is in τ.*

The members of τ are then called τ-open sets, or simply open sets. The pair (X, τ) is called a topological space. A subset A of a topological space (X, τ) is called a closed set if its complement A^c is an open set.

Definition 1.1.2 *[82] Let (X, τ) be a topological space and $A \subseteq X$. Then,*

1. *$cl(A) = \cap\{F \subseteq X : A \subseteq F$ and F is closed$\}$ is called the τ-closure of A,*
2. *$int(A) = \cup\{G \subseteq X : G \subseteq A$ and G is open$\}$ is called the τ-interior of A,*

3. $b(A) = cl(A) \setminus int(A)$ *is called the* τ*-boundary of* A.

Definition 1.1.3 *[82] Let* (X, τ) *be a topological space and* $x \in X$ *be an arbitrary point. A set* $N \subseteq X$ *is called a neighborhood of* x *if* $x \in int(N)$*, or equivalently, if there exists an open set* U *such that* $x \in U \subseteq N$.

Definition 1.1.4 *[37] Two subsets* A *and* B *of a topological space* (X, τ) *are said to be separated from each other in* X *if and only if* $cl(A) \cap B = A \cap cl(B) = \phi$.

Definition 1.1.5 *[70] A non-empty collection* I *of subsets of a set* X *is called an ideal on* X*, if it satisfies the following conditions:*

1. *if* $A \in I$ *and* $B \in I$*, then* $A \cup B \in I$,

2. *if* $A \in I$ *and* $B \subseteq A$*, then* $B \in I$,

i.e. I *is closed under finite unions and inclusions.*

Definition 1.1.6 *[72] Let* (X, τ) *be a topological space and* $A \subseteq X$*. Then,* A *is said to be a generalized closed (g-closed, for short) set if* $cl(A) \subseteq O$ *whenever* $A \subseteq O$ *and* O *is an open set.*

Definition 1.1.7 *[25] Let* R *be a binary relation on* X*. Then,* R *is called:*

1. *an identity relation* $\Leftrightarrow R = \{(x, x) : x \in X\}$,

2. *a reflexive relation* $\Leftrightarrow (x, x) \in R, \forall x \in X$,

3. *a symmetric relation* $\Leftrightarrow (x, y) \in R$ *implies* $(y, x) \in R \; \forall x, y \in X$,

4. *a transitive relation* $\Leftrightarrow (x, y), (y, z) \in R$ *implies* $(x, z) \in R \; \forall x, y \in X$,

5. *an equivalence relation* $\Leftrightarrow R$ *is reflexive, symmetric, and transitive relation,*

6. *a preorder relation* $\Leftrightarrow R$ *is reflexive and transitive relation,*

7. *an antisymmetric* $\Leftrightarrow$*(x, y)* $\in R$ *and* $(y, x) \in R$ *imply* $x = y$, $\forall \; x, y \in X$,

8. *a partial order relation if it is reflexive, antisymmetric and transitive.*

Definition 1.1.8 *[25] Let* U *be any set and* R *be any binary relation on* U*. The after set (respectively fore set) of the element* $x \in U$ *is the set* $xR = \{y \in U : xRy\}$ *(respectively* $Rx = \{y \in U : yRx\}$*).*

Definition 1.1.9 *[35] A function $f : (X, \tau) \to (Y, \theta)$ is called:*

1. *continuous if the inverse image of every open subset of Y is an open subset of X,*
2. *open (respectively closed) if the image of every open (respectively closed) subset of X is an open (respectively closed) subset of Y,*
3. *homeomorphism if f is one-to-one correspondence, continuous and open.*

1.2 Some generalizations of rough sets

In the following subsections, we collect Pawlak's approximation spaces [86] and several generalizations of those spaces [6, 62, 68]. Therefore, the relationships among them were given by [6, 62, 68, 117].

1.2.1 Pawlak's approximation space

Definition 1.2.1 *[86] Let R be an equivalence relation on a universe X, $[x]_R$ be the equivalence class containing x. For any subset A of X, the lower approximation $\underline{R}(A)$ and the upper approximation $\overline{R}(A)$ are defined by:*

$$\underline{R}(A) = \{x \in X : [x]_R \subseteq A\}, \tag{1.1}$$

$$\overline{R}(A) = \{x \in X : [x]_R \cap A \neq \phi\}. \tag{1.2}$$

Theorem 1.2.1 *[120] The upper approximation, defined by 1.2, has the following properties:*

1. $\overline{R}(\phi) = \phi$,
2. $A \subseteq \overline{R}(A)$, $\forall A \subseteq X$,
3. $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$, $\forall A, B \subseteq X$,
4. $\overline{R}(\overline{R}(A)) = \overline{R}(A)$, $\forall A \subseteq X$,
5. $\overline{R}(A) = (\underline{R}(A^c))^c$, $\forall A \subseteq X$.

Corollary 1.2.1 *[62] Let R be an equivalence relation on X. Then, the operator $\overline{R}$ on $P(X)$ defined by 1.2 satisfied the Kuratowskis axioms and induced a topology on X denoted by τ_R and defined as*

$$\tau_R = \{A \subseteq X : \overline{R}(A^c) = A^c\}. \tag{1.3}$$

1.2.2 Yao's approximation space

Definition 1.2.2 *[117] Let R be a binary relation on X and A be a subset of X. Then, the pair of lower and upper approximations, $\underline{R}(A)$ and $\overline{R}(A)$, are defined by:*

$$\underline{R}(A) = \{x \in X : xR \subseteq A\}, \quad (1.4)$$
$$\overline{R}(A) = \{x \in X : xR \cap A \neq \phi\}. \quad (1.5)$$

Theorem 1.2.2 *[8] If R is a preorder relation on X (i.e. R is a reflexive and a transitive relation on X), then the upper approximation, defined by 1.5, satisfies the properties in Theorem 1.2.1.*

1.2.3 Allam et al.'s approximation space

Definition 1.2.3 *[7] Let R be any binary relation on X, a set $< p > R$ is the intersection of all after sets containing p, i.e.,*

$$< p > R = \begin{cases} \cap_{p \in xR} xR, & \text{if } \exists\, x : p \in xR; \\ \phi, & \text{otherwise.} \end{cases}$$

Also, $R < p >$ is the intersection of all fore sets containing p, i.e.,

$$R < p >= \begin{cases} \cap_{p \in Rx} Rx, & \text{if } \exists\, x : p \in Rx; \\ \phi, & \text{otherwise.} \end{cases}$$

Definition 1.2.4 *[6] Let R be a binary relation on X. For any subset A of X, a pair of lower and upper approximations, $\underline{R}(A)$ and $\overline{R}(A)$, are defined by:*

$$\underline{R}(A) = \{x \in X :< x > R \subseteq A\}, \quad (1.6)$$
$$\overline{R}(A) = \{x \in X :< x > R \cap A \neq \phi\}. \quad (1.7)$$

Lemma 1.2.1 *[7] For any binary relation R on X if $y \in< x > R$, then $< y > R \subseteq< x > R$.*

Theorem 1.2.3 *[6] Let R a reflexive relation on X. Then, the upper approximation, defined by 1.7, satisfies the properties in Theorem 1.2.1.*