شبكة المعلومات الجامعية
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بسم الله الرحمن الرحيم

MONA MAGHRABY

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جامعة عين شمس

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MONA MAGHRABY

Geometrical aspects of Banach spaces and generalized projection methods

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"BESM ELLAH ERRAHMAN ERRAHEEM."

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"No One Beyond Mistake"

Contents

Summary

This Ph. D. thesis is organized as follows:

1. In chapter #1, we introduce a brief history and a motivation for the problem of approximating a fixed point, assuming that it exists, for single-valued mappings in Banach and Hilbert spaces and multi-valued mappings in Banach spaces and how this problem was solved by using the notions of both the metric and generalized projection operators. We explain the importance of generalized projection operator of Banach spaces that it was presented analogously to metric projection in Hilbert spaces.

2. In chapter #2, we present almost of the details needed in this thesis and it contains the most important definitions, examples, theorems and results obtained in various Banach spaces.

3. In chapter #3, we show the basic properties about various types of smoothness and convexity conditions that the norm of a Banach space may(or may not) satisfy. We present the normalized duality mapping of Banach spaces and explain the main role of it to determine the geometric properties of Banach spaces. Also, we introduce the concepts of Birkhoff orthogonality and J-orthogonality in Banach spaces and study their properties.

4. In chapter #4, we present orthogonality, projection methods in Hilbert spaces and clarify the relations between them. We introduce the metric projection operator and its properties in Banach spaces and explain the main links between metric projection and normalized duality mappings and the relation between metric projection and orthogonality in Banach spaces. We show the generalized projection operator and explain the generalization of it from uniformly convex and uniformly smooth Banach

spaces to reflexive Banach spaces. We also give the basic properties of generalized projection operators in uniformly convex and uniformly smooth Banach spaces and in reflexive Banach spaces.

5. In chapter #5, we introduce our new results, the first new result is published in Journal of Inequalities and Applications see [36], through this paper we generalize the concepts of normalized duality mappings, J-orthogonality and Birkhoff orthogonality from normed spaces to smooth countably normed spaces. Also, We give some basic properties of J-orthogonality in smooth countably normed spaces and we show the relation between J-orthogonality and metric projection on smooth uniformly convex complete countably normed spaces. Moreover, we define the J-dual cone and J-orthogonal complement on a nonempty subset S of a smooth countably normed space and we prove some basic results about the J-dual cone and the J-orthogonal complement of S. The second new result is submitted to The Illinois Journal of Mathematics and to be accepted soon see [37], through this paper we extend the concept of generalized projection operator "$\Pi_K : E \to K$" from uniformly convex uniformly smooth Banach spaces to uniformly convex uniformly smooth countably normed spaces and study its properties. Also, we show the relation between J-orthogonality and generalized projection operator Π_K and give examples to clarify this relation. Moreover, we introduce a comparison between metric projection operator P_K and generalized projection operator Π_K in uniformly convex uniformly smooth complete countably normed spaces in addition to extend the generalized projection operator "$\pi_K : E^* \to K$" from reflexive Banach spaces to uniformly convex uniformly smooth countably normed spaces and showing its properties.

Chapter 1

Introduction

1.1 Motivation

It is well known that, among all infinite dimensional Banach spaces, Hilbert spaces have the nicest geometric property structures. The fact that proximity map of a real Hilbert space H onto a closed convex subset K of H is nonexpansive; the polarization identity:

$$\|x+y\|^2 = \|x\|^2 + 2 < x, y > + \|y\|^2 \quad (i)$$

and the parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad (ii)$$

both hold for all $x, y \in H$. Consequently, to extend some of the Hilbert space techniques to more general Banach spaces, analogues of the identities (i) and (ii) have to be developed.

In this thesis we focus on the geometric properties of Banach spaces which are central in mathematical analysis.

Let K be a nonempty subset of a Banach space X. Let $f : K \to X$ be a map. The question of the existence and approximation of a fixed point is well known and important in functional analysis, well known iterative sequences for approximating fixed points of self maps may fail to be well defined. In the case that X is a Hilbert space, and K is a nonempty closed convex subset of X, one method of circumventing this problem is the introduction of the metric projection operator $P_K : H \to K$ into the recursion formula. For real

Hilbert space H , $K \subset H$ closed convex and $f : K \to H$, this problem has been resolved by using recurrence relation $x_{n+1} = P_K f x_n$, where P_K is the metric projection of H onto K. This is possible because we have that

$$\|P_K x - P_K y\| \leq \|x - y\| \quad \forall x, y \in H.$$

This fact is not available in general Banach spaces. In fact, P_K is nonexpansive if and only if the space is Hilbert space.

Suppose now that X is a real uniformly convex Banach spaces, $K \subset X$ is a closed convex set, it is well known that the metric projection $P_K : X \to K$ exists and is unique. If $f : K \to X$ is a mapping such that $F(f) \neq \emptyset$ where $F(f)$ is the set of fixed points of f, the recurrence relation,

$$x_{n+1} = f x_n, \quad n \geq 0 : \; x_0 \in K$$

may not be well defined. One method might attempt to solve this problem is to introduce the metric projection operator P_K and consider the recurrence relation

$$x_0 \in K : \; x_{n+1} = P_K f x_n, \quad n \geq 0.$$

Unfortunately, P_K does not have good properties in Banach spaces.

Recently, Alber [3] introduced an operator which is a generalization of the metric projection operator that is the generalized projection operator Π_K which has nice properties in general Banach spaces see [7].

It is our purpose in this project to study the properties of this operator. In particular we shall consider operators $f : K \to X$ where K is a closed convex subset of uniformly convex and uniformly smooth Banach space X, with $F(f) \neq \emptyset$. Using the notion of generalized projection.

Let X be a real uniformly convex and uniformly smooth Banach space and K be a nonempty closed convex subset of X. The generalized projection operator $\Pi_K : X \to K$ introduced by Alber is defined as follows :

$$\Pi_K x = \hat{x} \quad : \quad \varphi(x, \hat{x}) = \inf_{y \in K} \varphi(x, y),$$

where $\varphi(x,y) = \|x\|^2 - 2 < jx, y > + \|y\|^2$ and $j(x) \in J(x)$, where $J : X \to 2^{X^*}$ is the normalized duality mapping which has become a most important tool plays a central role in nonlinear functional analysis. Alber proved that Π_K has the following important property

$$\varphi(\Pi_K x, \Pi_K y) \leq \varphi(x,y) \quad \forall x \in X,\ y \in K.$$

In this Ph. D. project, geometric characteristics of Banach spaces such as convexity and smoothness of spaces, duality mappings and projection operators will play a very important role in this thesis,e.g., Characterizations of real uniformly convex and uniformly smooth Banach spaces by means of continuity properties of the normalized duality mappings. Also, we introduce a new orthogonality concept, that is called a J-orthogonality in smooth Banach spaces, by using the normalized duality mapping which is equivalent to the Birkhoff orthogonality in Banach spaces, and give some basic properties of J-orthogonality in a smooth Banach spaces.

In our new results see [36] and [37] we introduce some geometric properties of countably normed spaces. We extend the concepts of normalized duality mappings, J-orthogonality and Birkhoff orthogonality from normed spaces to smooth countably normed spaces. We give some basic properties of J-orthogonality in smooth countably normed spaces and we show the relation between J-orthogonality and metric projection on smooth uniformly convex complete countably normed spaces. Moreover, we define the J-dual cone and J-orthogonal complement on a nonempty subset S of a smooth countably normed space and we prove some basic results about the J-dual cone and the J-orthogonal complement of S. We extend the concept of generalized projection operator "$\Pi_K : E \to K$" from uniformly convex uniformly smooth Banach spaces to uniformly convex uniformly smooth countably normed spaces and study its properties. We show the relation between J-orthogonality and generalized projection operator Π_K and give examples to clarify this relation. We present a comparison between metric projection operator P_K and generalized projection operator Π_K in uniformly convex uniformly smooth complete countably normed spaces, we give an example explaining how to evaluate metric projection P_K and generalized projection Π_K in some cases of countably normed spaces and this example prove that the generalized projection operator Π_K in general, is set-valued mapping. Also we extend the generalized projection operator "$\pi_K : E^* \to K$" from reflexive Banach spaces to uniformly convex uniformly smooth countably normed spaces and showing its properties.

Chapter 2

Preliminaries

2.1 Basic definitions (concepts)

Definition 1 (Weak topology) ([14, 35, 38])
Let X be a Banach space. For each $f \in X^*$, we associate a map $\phi_f : X \to \mathbb{R}$ defined by $\phi_f(x) = f(x) \;\; \forall x \in X$. As f ranges over X^*, we obtain a family $\{\phi_f\}$ of functionals from X into $\mathbb{R}$. The *weak topology* on X is the smallest topology on X which makes the maps ϕ_f continuous.

The topology of infinite dimensional spaces is too big to be able to give us compactness. Thus, in order to obtain some form of compactness it is necessary to cut down the number of open sets under consideration, i.e., it is necessary to reduce the size of the topology of the infinite dimensional space X to obtain the weak topology on X in which we have what can be regarded as a generalization of the Heine-Borel Theorem to infinite dimensional spaces given by Kakutani.
Open respectively closed sets in the weak topology are also open respectively closed in the strong topology. This is easy to see since the strong topology contains more open sets than the weak. The converse, is not always true as in the following examples:

Example 2 ([14, 39, 40])
The set $S = \{x \in X : \|x\| = 1\}$ is clearly closed in the strong topology but it is not closed in the weak topology. For; let $x_0 \in X, \|x_0\| < 1$, i.e., $x_0 \notin S$. Let N be an arbitrary neighborhood of x_0

in the weak topology, suppose that

$$N = \{x \in X : | < f_i, x - x_0 > | < \epsilon, i = 1, ..., n\},\ \epsilon > 0,\ f_i \in X^*.$$

Fix

$$y_0 \in X, y_0 \neq 0,\ < f_i, y_0 >= 0,\ i = 1, ..., n.$$

Define the function $g : [0, \infty) \to \mathbb{R}$ by $g(t) = \|x_0 + ty_0\|$. Then g is continuous on $[0, \infty)$, $g(0) < 1$, $\lim_{t\to\infty} g(t) = \infty$.
Then, by the Intermediate Value Theorem, there exists $t_0 > 0$ such that $1 = g(t_0) = \|x_0 + t_0y_0\|$. This implies that $x = x_0 + t_0y_0 \in S$. Observe that

$$< f_i, x - x_0 >=< f_i, t_0y_0 >= t_0 < f_i, y_0 >= 0 < \epsilon,$$

so that $x = x_0 + t_0y_0 \in N$, i.e., $N \cap S \neq \emptyset$. Since N is an arbitrary neighborhood of x_0 in the weak topology, then we have proved that $x_0 \in clS_w$ (the closure of S in the weak topology). Thus $clS_w \neq S$.

Example 3 ([14])
The set $O = \{x \in X : \|x\| < 1\}$ is open (since its complement is closed) in the strong topology but it is not open in the weak topology since any neighborhood in the weak topology of any point in O intersects O but is not included in O because it contains the point $x_0 + t_0y_0$ which is not in O.

Definition 4 (The weak star topology) ([14])
For each $x \in X$ we consider the map $\phi_x : X^* \to \mathbb{R}$ defined by $\phi_x(f) = f(x)$. As x ranges over X we obtain a family of maps $\{\phi_x\}_{x\in X}$ from X^* into $\mathbb{R}$. The *weak star topology* w^* is the smallest topology on X^* for which all the maps ϕ_x are continuous.

Definition 5 (Weak convergence) ([14])
Let (X, w) be a Banach space endowed with the weak topology, and let $\{x_n\}$ be a sequence in X, then $x_n \rightharpoonup x$ *weakly* in the sense of w if and only if $f(x_n) \to f(x)\ \forall f \in X^*$.

Definition 6 (Weakly sequentially complete) ([14])
A Banach space X is said to be *weakly sequentially complete* if every weakly convergent sequence in X converges weakly to an element of X. That is, if $\{x_n\} \subset X$ are such that $\lim_{n\to\infty} f(x_n)$ exists for every $f \in X^*$, i.e.,

$$\exists\ x \in X; f(x) = \lim_{n\to\infty} f(x_n)\ \forall f \in X^*.$$