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Different Approaches for Solving Fractional Differential Equations

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Contents

Summary

This M.Sc thesis is organized as follows:

1. **Introduction**, we present a brief history of fractional operators and we explain the importance of Sinc method then introduce our aim of thesis.

2. **In chapter 1**, we present the basics needed in thesis and it contains the most important definitions, examples and theorems, where we explain some basics of Sinc function, conformal map, Sinc basis, Sinc approximation and singular points.

3. **In chapter 2**, we make a survey on some numerical methods which used to solve the differential equations, these methods are under the titles finite element methods, Spectral methods, finite difference methods and Sinc method and we state some properties of Sinc function with proofs, and introduce different definitions of fractional operators introduced by Riemann - Liouville and Caputo, then state fractional differential equations, and we explain how to convert them to convolution integrals by the definition of Riemann - Liouville, also we discuss Sinc method to solve indefinite and convolution integrals and then review the solution of fractional differential equations.

4. **In chapter 3**, we solve some applications of fractional differential equations with a fractional operator and with several fractional operators by using Sinc method and compare our results with the pervious results from other methods, also we use Thieles algorithm for approximation. The new result is already **accepted** for publication under title
"Fractional Differential Equations Solved by Convolution Integrals"
in the journal
"Italian Journal of Pure and Applied Mathematics, number 44, 2020, ISSN 2239-0227", see [41].

5. **In chapter 4**, we present our next future work in Ph.D work or other research for solving fractional differential equations.

6. **Conclusion**, we present the advantages and disadvantages of the Sinc method through the results we obtained in chapter 3.

7. **Bibliography**, we present all references used and referred to in the thesis.

Introduction

Fractional calculus is one of the oldest calculi invented. It dates back to Lipinze, in 1695 Lipinze asked L'Hopital: what about if we extend the order of derivative to half? which is fractional order. Lipinze said that if this question is answered, then it will open many consequences in different applications. Later, after about a hundred years, i.e., in 1819, Lacroui answered the question and initiated the following formula for every $q \in \mathbb{Q}$:

$$\frac{d^q}{dx^q}\, x^m = \frac{\Gamma(m+1)}{\Gamma(m-q+1)}\, x^{m-q}$$

We known that

$$\frac{d^n}{dx^n}\, x^m = \frac{m!}{(m-n)!}\, x^{m-n}$$

for every $n \in \mathbb{N}$. But for n is fraction, the vectorial $(m-n)$ become vectorial of fraction number but the vectorial is defined only for natural numbers. However, the gamma function is defined for fraction numbers and there is a relation between gamma function and vectorial so that he replace the vectorial by gamma function.

Interesting point that the fractional derivative D^α for 1 is not equal to zero but equal to $\frac{x^{-\alpha}}{\Gamma(1-\alpha)}$ (by using Riemann-Liouville definition of fractional derivative), while $D^n(1) = 0$ for every $n \in \mathbb{N}$

Subsequent mention of fractional derivatives was made, in some context or the other, by (for example) Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Griinwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917.

In addition, of course, to the theories of differential, integral, and integro-differential equations, and special functions of mathematical physics as well as their extensions and generalizations in one and more variables, some of the areas of present-day applications of fractional calculus include Fluid Flow, Rheology, Dynamical Processes in Self-Similar and Porous Structures, Diffusive Transport Akin to Diffusion, Electrical Networks, Probability and Statistics, Control Theory of Dynamical Systems, Viscoelasticity, Electrochemistry of Corrosion, Chemical Physics, Optics and Signal Processing, and so on. So all science-topics based on integer derivative could be generalized to the fractional calculus form. For an historical overview on fractional calculus, see [1, 26, 27].

There are a lot of numerical methods used for approximation, but most of these methods do not approximate all types of functions but deal with specific functions, for example: polynomials approximation deals with analytic functions without singularities, and Fourier polynomials approximation is applicable to functions that are smooth and periodic on some domains. So, we follow a new direction of approximation theory "Sinc method" introduced by Stenger in connection with convolution integrals [9]. The theoretical background of the method can be found in [10]. We will use Sinc approximation because it solves problems with singularities, boundary layer problems, and problems over infinite and semi infinite domains. Furthermore, the computer programs based on Sinc methods are usually considered shorter than the corresponding ones based on classical methods of approximation. Sinc methods are particularly powerful when the problem has singularities, that means, in case of the error of an n-point approximation converges at an incredible $O(e^{-cN^{\frac{1}{2}}})$ rate, whereas in such circumstances polynomial methods, at best, converge at the $O(n^{-a})$ rate, where c and a are positive constant [10, 29], where

Notation	Name	Description
$f(n) = O(g(n))$	Big O	$\|f\|$ is bounded above by g asymptotically
$f(n) = o(g(n))$	Small o	f is dominated by g asymptotically
$f(n) = \Theta(g(n))$	Big Theta	f is bounded both above and below by g asymptotically
$f(n) = \Omega(g(n))$	Big Omega in complexity theory	f is bounded below by g asymptotically
$f(n) = \Omega(g(n))$	Big Omega in number theory	$\|f\|$ is not dominated by g asymptotically
$f(n) = w(g(n))$	Small Omega	f dominates g asymptotically
$f(n) \sim (g(n))$	On the order of	f is equal to g asymptotically

Oftentimes the true value is unknown to us, especially in numerical computing. In this casewe will have to quantify errors using approximate values only. When an iterative method is used, we get an approximate value at the end of each iteration. The approximate error E_α is defined as the difference between the present approximate value and the pervious approximation (i.e. the change between the iterations).
Approximate error = Present approximation - Pervious approximation

Similarly, we can calculate the relative approximate error e_α by dividing the approximate error by the present approximate value.
Relative approximate error = Approximate error / Present approximation.

This thesis deals with Sinc method via convolution integrals for solving fractional differential equations. We tested examples and compared them with exact solutions, so it is shown that Sinc method yields accurate results. Also we use the elliptic function procedure to derive a family of interpolating rational approximations. The rationals which we shall construct have accuracy equivalent to that of Sinc approximation, in the spaces which we have shown Sinc methods to be effective. We may expect Thieles algorithm to yield an accurate approximation of a function G at a boundary point b of its region of analyticity. By accurate, we mean that using O(N) values of G, we can approximate $G(b)$ to be within an O($e^{-CN^{\frac{1}{2}}}$) error, with C a positive constant, depending for practical purposes neither on N, nor on the particular set of given data $\{w_j, G(w_j)\}$ [1, 11].

The aim of this thesis is to introduce a new approach to represent fractional operators based on Sinc method. This approach has the advantage that we need only a single base of functions allowing us to represent approximations of functions by a truncation of a cardinal expansion [26, 29]. In addition, the derived approximation is not only numerically exponentially converging and can handle singularities but also provides symbolic access to the results which in combination with numeric delivers a hybrid representation of the approximation. We will discuss this approach on a specific selection of fractional differential operators.

Chapter 1

Preliminaries

In this chapter, we give some basics needed for the research presented in this Master thesis. In section 1.1 and 1.2, we start with a review of conformal maps leading to the definition of Sinc points. In section 1.3 and 1.4, we discuss the Sinc approximations tools and properties and Sinc basis. In section 1.5, we introduce the notation of singular points. The background for applications in chapters 3 is not covered in this chapter but in chapters 2.

1.1 Conformal Map

Conformal mapping is a mathematical tool which by complicated geometries can be transformed into simpler geometries that still preserve both the angles and orientation of original geometry [20]. For example, when this tool is used, the fluid flow around the geometry of an airfoil can be analyzed as the flow around a cylinder whose symmetry simplified the needed computation [23]. This concept of transformation will be used in the interpolation theory as follow: given a problem on an interval and given a numerical process defined on the real line, the effective procedure to solve this problem is to transform the numerical process from real line to the tested interval and to study it on the new domain. The development for transforming these methods from one domain to another is accomplished via conformal mappings.

The existence of a conformal mapping for a specified domain $\mathcal{D}$ onto another domain $\mathcal{G}$ is attributed to Riemann and can be found in Lars V. Ahlfors [30].

Definition 1.1 Simply Connected Domain[11, 30]
A region $\mathcal{D}$ is said to be simply connected domain if any simple closed curve that lies entirely in $\mathcal{D}$ can be pulled to a single point in $\mathcal{D}$ (a curve is called simple if it has no self-intersections).

Definition 1.2 Riemann Mapping Theorem[11, 30, 36]
If $\mathcal{D}$ is any simply connected domain in the plane (other than the entire plane itself), then there exists a one-to-one conformal mapping $w = f(z)$ that maps $\mathcal{D}$ onto the unit disk $|w| < 1$.

Definition 1.3 Analytic Function[36]
A complex function $f(z)$ is said to be analytic on a region $\mathcal{D}$ if it is complex differentiable at every point in $\mathcal{D}$.

Definition 1.4 Conformal Mapping[36]
Let $\mathcal{D} \subset \mathbb{C}$ and $f : \mathcal{D} \to \mathbb{C}$. Let z_0 be any point of $\mathcal{D}$. Then, f is called a conformal mapping at z_0 if f is complex differentiable at z_0 and $f'(z_0) \neq 0$.

The geometric meaning of the last definition is that a conformal mapping is a transformation $w = f(z)$ that perserves the local angles. In more detail, if the mapping $w = f(z)$ is conformal at z_0, then the angle between two arcs through z_0 is the same as the angle between their images under f at $f(z_0)$; the orientation of the angle is , thus, preserved.

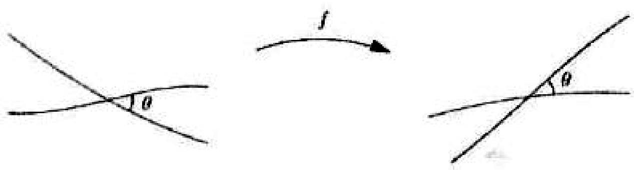

figure 1.1: Conformal Map

In the thesis, some conformal mappings are used to transform a certain approximation from the real line to a given interval. Based on the choice of these intervals, the conformal mappings are selected. Next, we introduce examples of these conformal mappings used in this thesis.