Ain Shams University
Faculty of Science
Mathematics Department

On Generalizations of Continuous Modules

Thesis

Submitted By

Manar El-Badry Tabarak Ebrahim

Assistant Lecturer, Mathematics Department, Faculty of Science
Ain Shams University

For the Degree of
Doctor of Philosophy in Science
(Pure Mathematics)

Supervised By

Prof. Dr.
Salah El-Din Sayed Hussein (Dead)
Professor Emeritus of Pure Mathematics
Mathematics Department
Faculty of Science- Ain Shams Uni.

Prof. Dr.
Essam Ahmed Soliman El-Seidy
Professor of Pure Mathematics
Mathematics Department
Faculty of Science- Ain Shams Uni.

Dr.
Mahmoud A. Kamal El-Deen (Dead)
Associate Emeritus Professor of Pure Mathematics
Mathematics Department
Faculty of Education - Ain Shams Uni.

Dr.
Maha H. Ebrahim El-Baroudy
Lecturer of Pure Mathematics
Mathematics Department
Faculty of Education - Ain Shams Uni.

Submitted to
Mathematics Department, Faculty of Science
Ain Shams University
Cairo Egypt

2020

Table of Contents

Acknowledgement

I humbly acknowledge the blessings of **Almighty, Compeller and Subdeur Allah**, without **Whose** mercy and guidance this work would have never started nor completed. May **Allah** pray on Mohamed ”Peace Be Upon Him” the prophet and the Massenger of **Allah**.

I would like to express my deep gratitude to **Prof. Dr. Salah El-Din Sayed Hussien**, (May Allah have mercy on him), Professor Emeritus of Pure Mathematics, Department of Mathematics, Faculty of Science, Ain Shams University, for his huge support, patience, care, criticism, advice and much help. I do not forget his advice on writing the thesis and his careful review of a large part of it. He has always encouraged and helped me to participate in valuable seminars ”May Allah reward him the best”.

I would like to thank **Prof. Dr. Essam Ahmed Soliman El-seidy**, Professor of Pure Mathematics, Department of Mathematics, Faculty of Science, Ain Shams University, for his help, support, and for facilitating every thing for me, giving so generously of his time to complete this thesis in its final form.

I wish to record my deep gratitude to **Dr. Mahmoud Ahmed Kamal El-Deen**,(May Allah have mercy on him), Associate Emeritus Professor of Pure Mathematics, Department of Mathematics, Faculty of Education, Ain Shams University, for being wonderful mentor throughout my years in preparing my thesis by suggesting points of research, background material, actively collaborating in much of my work, and for his inspiration, support, patience, care, criticism, advice, encouragement and great efforts to explain things clearly and simply and for his guidance throughout this work ”May Allah reward him the best”.

My special thanks are due to **Dr. Maha Helmy Ebrahim El-Baroudy**, Lecturer of pure Mathematics, Department of Mathematics, Faculty of Education, Ain Shams University, for her continuous guidance throughout this work and for her huge help, patience, care, criticism, encouragement, and sincere advice. I am grateful for her careful review of the thesis and for giving so generously of her time to complete this thesis in its final form.

Also, I would like to thank all Algebra's seminar members in Department of Mathematics Faculty of Education **Dr. Mahmoud A. Kamal, Dr. Maha H. E. El-Baroudy, Dr. Nawal Sayed Mahmoud, Dr. Ahmed Youssef, Dr. Atef Sayed El-Araby and Mahmoud saad Mehany** for their help, discussion and explaining the basic knowledge in a friendly manner. They have all been very helpful and have shared a lot of time with me during my work. I consider myself lucky to work closely with all of them.

I wish to record my gratitude to the staff of Department of Mathematics, Faculty of Science, Ain Shams University, for giving me opportunity to pursue my studies.

Many thanks also go to all my colleagues, in my department who wished for me a good luck, wishing for them all the best in their life. And special thank go to my close friends for their encouragement, help, and kind continuous support.

Finally, I owe deep thanks to my family, I offer my sincere gratitude for their love and continuous support and for giving me the opportunity to pursue my education. And I am honored to dedicate this work to the memory of my late parents (May Allah have mercy on them).

Abstract

The main objective of this thesis is twofold. First, we aim to introduce the concept of weak large extensions for modules as a general case of the notion of large extensions. Some properties of large extensions hold true for weak large extensions, while some others need special types of modules (e.g. non-singular modules), and special types of rings. Weak large extending modules (modules in which every WL-closed submodule is a direct summand) are also introduced here. We show that, as of the case of extending modules, the second singular submodule of a *W*L-extending module splits. Second, we aim to introduce and investigate the concept of relative superfluous injective modules and weak superfluous injective modules. On the other hand we introduce and characterize the concepts of superfluous extending (for short S-extending) modules, weak superfluous extending (for short WS-extending) modules and superfluous quasi-continuous(for short S-quasi-continuous) modules. We prove that essential properties of injective modules still hold true for superfluous injective modules and weak superfluous injective modules. We also establish a convenient characterization for S-extending, WS-extending and S-quasi-continuous modules of non-zero radicals.

Summary

Throughout this thesis R will denote an associative ring with unit and all modules are unital right R-modules. We use $L \leq M$ and $N \leq^{\oplus} M$ to denote that L is a submodule and N is a direct summand of M, respectively.

An effective way to understand the behaviour of a ring R is to study the various ways in which R acts on its left and right R-modules. Thus, the theory of modules can be expected to be an essential chapter in the theory of rings. The study of rings and modules would be deficient without a connection with homological algebra. In the solar system of homological algebra, the sun is certainly the theory of projective and injective modules.

The importance of injective modules in module theory and more generally in algebra became obvious in the 1960s and 1970s largely, but not exclusively, through the impact of the publication of the lecture notes of Carl Faith [9]. Since that time there has been a continuing interest in such modules and their various generalizations which arose not only directly from the study of injectives but also from work of John von Neumann concerning his attempt to model Quantum Mechanics via continuous geometries. Classes of generalizations of injective modules including quasi-injective, continuous and quasi-continuous modules.

The concept of relative injectivity of modules has been generalized by many authors, this has been achieved by adding extra conditions on the homomorphisms in the defining diagram. Generalizations of relative injectivity have been widely investigated and many important connections between these generalizations have been established(cf. [5], [28], [48]).

The continuity of modules, as the term is used here, is not related to continuity in the sense of topology and analysis. It is rather derived from the notion of a continuum. This usage originated with von Neumann's continuous geometries.

Utumi observed that continuous regular rings generalize self-injective regular rings [45]. Then he extended the concept to arbitrary rings [46]. Jeremy, Mohamed and Bouhy, and Goel and Jain generalized these ideas to modules (cf. [15], [30], [10]). In connection with quasi-Frobenius rings, M. Yousif [47] has studied right continuous rings.

The continuous and quasi-continuous modules have a common property which is a generalization of injectivity namely, the extending property for submodules i.e. every submodule is large in a direct summand. Actually, the importance of the study of extending modules has been attracting a great interest from many authors in the 1980s, including M. Harada and his school and B. Miiller with various collaborators (cf. [39], [31], [33], [34], [32], [21], [20], [19], [12], [6]). More recently, the study of extending modules has been given a major boost by the observation that the extending property for all cyclic subfactors of a module M implies finite uniform dimension of all cyclic, or even finitely generated, subfactors of M. This result has come to be known as the Osofsky-Smith Theorem.

During the past thirty years the concept of extending modules and its generalizations have been investigated intensively and extensively (cf. [1], [18], [21] [20], [22], [16]).

Although, this generalization of injectivity is very useful, it does not satisfy some important properties. One of the main important properties is, when do the direct sum of modules with a property P inherit this property?. It is known that this property does not hold true in general even for finite direct sums of extending modules.

Many authors have studied this problem to find the necessary and sufficient conditions to ensure that the extending property and its generalizations are preserved under various extensions.(cf. [17], [19], [20], [23], [17], [35]).

The main objective of this thesis is twofold. First, we aim to introduce and investigate thoroughly the concept of weakly large submodules as a generalization of large submodules and use this concept to generalize extending property for modules. We introduce the concept of weakly large extending modules and study them deeply. Second, we aim to use superfluous submodules to define Con-S-submodules, semi-strict and strict Con-S-submodules. We use these types of submodules to introduce generalizations of relative injectivity and extending property for modules. Actually, we introduce the concepts of relatively superfluous injectivity, relatively weak superfluous injectivity, S-extending and weak S-extending modules. We also study what are the necessary and sufficient conditions to ensure that the finite direct sum of modules which are S-extending (weak S-extending) is S-extending (weak S-extending). The new results we have obtained so far are presented in chapters 2, 3, and 4.

This thesis consists of four chapters. In what follows we give a brief coverage of the contents of the thesis.

In the first chapter we assemble the basic concepts, definitions, and associated well known results which are necessary for our study.

The second chapter consists of four sections. For the sake of comparison we start in the first section by presenting the definitions and some properties of the strongly large submodules and strongly large extending modules (cf. [44], [43]). In the second and third sections, we introduce and investigate the concepts of weakly large submodules and weakly large closed submodules. Actually, we examine homomorphic images, inverse homomorphic images, direct sums, direct summands, intersections and factor modules of weakly large and weakly large closed submodules. In section four, we use the concept of weakly large closed submodules to introduce the concept of weakly large extending modules. One of the main results of this section is (Theorem 2.4.16). In which we prove that a module M, with $Z(M) \lneqq M$ is weakly large extending if and only if $M = N \oplus Z(M)$, with $Z(M)$ (a WL-uniform submodule) is weakly large extending, and N is non-singular large extending. We round off by deducing a characterization of weakly large extending modules over a right noetherian ring in terms of weakly large uniform submodules for which every local direct summand is a direct summand (Theorem 2.4.20).

The third chapter is divided into three sections. In the first section, we confine our attention to study a generalization of relative injectivity by defining Con-S-submodules. One of the main results of this secion is (Theorem 3.1.17), which is an important characterization of superfluous injective modules relative to a module N. In the second section, we introduce the concept of S-extending modules as a generalization of extending modules. For the sake of completeness of our study we devote the third section to study the necessary and sufficient conditions for the finite direct sum of S-extending modules to be S-extending. In (Theorem 3.3.9 and Theorem 3.3.11), we make use relatively superfluous injectivity to prove that a finite direct sum of S-extending modules is S-extending.

The fourth chapter is divided into three sections. In the first section, we introduce the concepts of semi-strict and strict Con-S-submodules. In section 2, as a generalization of relative superfluous injectivity, we introduce the concept of relative weak superfluous injectivity along with a thorough study of its properties. One of the main results of this section is (Theorem 4.2.11). In which we prove that M_2 is WS-injective relative to M_1 if and only if $M = C \oplus M_2$ holds, for every a strict Con-S-complement C of M_2 in M, whenever $M = M_1 \oplus M_2$; $Rad(M_i) \neq 0$, and $C \cap M_1 \leq^{Con-S} M_1$. This result is an important characterization of WS-injective modules relative to a module N. Also discuss when the classes of relative WS-injectivity and relative injectivity are coincide. The third section is devoted to the investigating and introducing the concept of weak S-extending modules, which is used to introduce the concept of S-quasi-continuous modules as a generalization of quasi-continuous modules. One of the main results of this section is (Theorem 4.3.16), which states that, a module M with a non-zero radical is S-quasi-continuous if and only if $M = L \oplus N$, for any two submodules L and N which are strict Con-S-complement of each other.

Two papers for the main results of chapters 2 and 3 of this thesis are accepted and published in [8] and [41].

The paper for the main results of chapter four of this thesis is submitted to **Italian Journal of Pure and Applied Mathematics** [40].

1

Preliminaries

In this chapter we introduce the background material which we need in this thesis. However, we just provide a brief survey of the basic definitions and elementary results concerning injective modules and some of its generalization, extending, continuous and quasi-continuous modules. For details on modules and injectivity of modules we refer to [24], [25], [9], [2], [14], [13], [11] and [4], while for details on extending, continuous and quasi-continuous modules we refer to [6], [31], [42], and [3].

1.1 Basic Definitions and Results

Large and Superfluous Submodules

Definition 1.1.1. *A submodule N of a module M is called a large submodule of M or M is a large extension of N (denoted by $N \trianglelefteq M$) if $N \cap L \neq 0$, for every $0 \neq L \leq M$ or equivalently, if $L \cap N = 0$ implies $L = 0$, for every submodule L of M.*

Definition 1.1.2. *A non-zero module M is called uniform if every non-zero submodule of M is large in M. We say that a module M has a uniform dimension n (written $u.dim(M) = n$) if there is a large submodule V of M which is a direct sum of n uniform submodules. On the other hand, if such integer n does not exist, we write $u.dim(M) = \infty$.*

For any submodule N of a module M and $m \in M$, we denote to the right ideal $\{r \in R : mr \in N\}$ of R by $m^{-1}N$.

Lemma 1.1.3. *([24], Lemma 5.1.6) Let N be a submodule of a module M, then the following are equivalent:*

1. *N is a large submodule of M.*
2. *For every $0 \neq m \in M$, there exists $r \in R$ such that $0 \neq mr \in N$.*
3. *For every $m \in M$, there exists a large right ideal I of R such that $mI \leq N$.*
4. *$m(m^{-1}N) \neq 0$ for all $0 \neq m \in M \setminus N$.*

The following proposition contains some facts about large submodules.

Proposition 1.1.4. *([2], Proposition 5.16) Let M be a module with submodules $L \leq N \leq M$, and $K \leq M$. Then*

1. *$L \trianglelefteq M$ if and only if $L \trianglelefteq N$ and $N \trianglelefteq M$.*

2. *$K \cap L \trianglelefteq M$ if and only if $K \trianglelefteq M$ and $L \trianglelefteq M$.*

3. *If $A \trianglelefteq N$ and $f : M \to N$ is a homomorphism, then $f^{-1}(A) \trianglelefteq M$.*

Proposition 1.1.5. *([2], Proposition 6.17) Let $\{N_i\}_{i\in I}$ be a set of independent submodules of a module M. If $\{M_i\}_{i\in I}$ is a set of submodules of M such that $L_i \trianglelefteq M_i$, for each $i \in I$, then*

1. *$\{M_i\}_{i\in I}$ is independent.*

2. *$\oplus_{i\in I} L_i \trianglelefteq \oplus_{i\in I} M_i$.*

Definition 1.1.6. *A submodule C of a module M is called a large closed submodule (denoted by $N \leq^C M$)of M if C has no proper large extension in M.*

Definition 1.1.7. *Let C and L be submodules of a module M. Then C is called a complement of L in M if it is maximal with respect to the property $C \cap L = 0$. Such submodule C always exists, by virtue of Zorns Lemma. In fact, any submodule C_0 of M satisfying $A \cap C_0 = 0$ can be enlarged to a complement of A.*

Proposition 1.1.8. *([11], Proposition 1.3) Let L and N be submodules of a module M such that $L \cap N = 0$. Then L is a complement of N in M if and only if L is a large closed submodule of M and $L \oplus N$ is large in M.*

Proposition 1.1.9. *([7], Lemma 1.23) If N' is a complement of N in a module M and N'' is a complement of N' in M then*

1. *N' is also a complement of N''.*

2. *If $N \leq N''$, then we have $N \trianglelefteq N''$*

Lemma 1.1.10. *([27], Lemma 1.33) Let N and N' be submodules of a module M such that $N' \trianglelefteq N$. Then C is a complement of N in M if and only if C is a complement of N' in M.*

Lemma 1.1.11. *([29], Lemma 2) Let L and N be submodules of a module M, with $L \leq N$. If C is a complement of L in M, then $C \cap N$ is a complement of L in N.*

Corollary 1.1.12. *([7], Corollary 3.22) Let N be a large closed submodule of a module M, and let C be a complement in M, for some submodule L of N. Then $C \cap N$ is a large closed submodule of M.*

Proposition 1.1.13. *([11], Proposition 1.4) Let N be a submodule of a module M. Then the following are equivalent:*

1. *N is large closed in M.*

2. *If $N \leq L \trianglelefteq M$, then $L/N \trianglelefteq M/N$.*

3. *N is a complement for some $L \leq M$.*

4. *If C is any complement of N in M, then N is a complement of C in M.*

Lemma 1.1.14. *([7], Lemma 4.4) Let $M = M_1 \oplus M_2$. Then C is a complement of M_2 in M if and only if C is the graph of a maximal partial homomorphism from M_1 into M_2.*

Lemma 1.1.15. *([7], Lemma 3.5) Let $M = \oplus_{i\in I} U_i$, where the U_i's are uniform. If A is a submodule of M, then there exists $J \subseteq I$ such that $\oplus_{i\in J} U_i$ is a complement of A in M.*

Proposition 1.1.16. *([11], Proposition 3.20)*

1. *If N is a large closed submodule of a module M, then $u.dim(M) = u.dim(N) + u.dim(M/N)$.*
2. *If $\{N_i\}_{i=1}^{n}$ is an independent family of submodules of a module M, then $u.dim(\oplus_{i=1}^{n} N_i) = \sum_{i=1}^{n} u.dim(N_i)$.*
3. *Let $N \leq M$ and assume that M is a finite-dimensional module. Then $u.dim(N) = u.dim(M)$ if and only if $N \trianglelefteq M$.*
4. *If N is ismorphic to M, then $u.dim(N) = u.dim(M)$.*

Definition 1.1.17. *A submodule N of a module M is called a superfluous submodule of M(denoted by $N \ll M$) if $N + K = M$ implies $K = M$, for every submodule K of M or equivalently, $N + K \neq M$, for any proper submodule K of M.*

Lemma 1.1.18. *([24], Lemma 5.1.3) Let M be a module, and $N \leq L \leq K \leq M$, then:*

1. *If $L \ll M$, then $N \ll M$.*
2. *If $N \ll L$, then $N \ll K$.*
3. *If $N \ll M$ and $L \leq^{\oplus} M$, then $N \ll L$.*