



شبكة المعلومات الجامعية  
التوثيق الإلكتروني والميكروفيلم

# بسم الله الرحمن الرحيم



**MONA MAGHRABY**



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# شبكة المعلومات الجامعية التوثيق الإلكتروني والميكروفيلم



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# جامعة عين شمس

## التوثيق الإلكتروني والميكروفيلم

### قسم

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### يجب أن

تحفظ هذه الأقراص المدمجة بعيدا عن الغبار



**MONA MAGHRABY**



Department: Mathematics

# Estimation of Stress-Strength Reliability for Some General Bivariate Distributions

A Thesis submitted in Partial Fulfillment of the  
requirement for the Master Degree in Science in  
Mathematical Statistics

By

Student name/ Dina Ahmed Mohamed Abd El-Razik

To

Department: Mathematics

Faculty of Science - Ain Shams University

Supervised by

Prof. Dr. Nahed Abd El Salam  
Mokhlis

Emeritus Professor of Mathematical  
Statistics-Department of  
Mathematics, Faculty of Science,  
Ain Shams University

Dr. Sohair Khames Khames  
Gomaa

Lecturer of Pure Mathematics-  
Department of Mathematics,  
Faculty of Women, Ain Shams  
University

Year(2021)



نموذج بيانات منح درجة  
قسم : الرياضيات

**Title Page**

**Name:** Dina Ahmed Mohamed Abd El-Razik

**Degree:** Master of Science in Mathematical Statistics

**Department:** Department of Mathematics

**Faculty:** Faculty of Science

**University:** Ain Shams University

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***For the soul of my beloved father.***

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## **Abstract**

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The main objective of this thesis is the point estimation of a stress–strength model,  $R = (X_1 > X_2)$ , in two cases: the first case when  $X_1$  and  $X_2$  are dependent variables with marginal distribution functions having the same forms of distribution general exponential or general inverse exponential forms. The second case is when  $X_1$  and  $X_2$  are dependent variables with marginal distribution functions having different forms of distribution general exponential and general inverse exponential forms or vice versa. Different point estimators are obtained by different methods which can be classified into two main methods, the classical point estimation method and the Bayesian method. The estimation is performed based on complete sample and double Type II censored sample which contain the left Type II censored sample, right Type II censored sample and complete sample as special cases. Various statistical distributions in the literature possess the general exponential and the general inverse exponential forms. So, the results obtained can be applied to all these distribution. As illustration of the results obtained, Weibull and generalized inverted exponential are applied as example of general exponential form. While Burr III and generalized exponential are applied as example of general inverse exponential form. Simulation studies are also carried out for comparison of the different point estimators obtained. The comparison is based on mean squared error and the bias of the different estimators. A real life data set is also presented to demonstrate the applicability of the forms studied and the results obtain.

**Keywords:** Farlie– Gumbel– Morgenstern copula, general exponential form, general inverse exponential form, stress–strength reliability model, Metropolis–Hastings algorithm, Type II censored data, maximum likelihood estimator, Bayesian estimator, MCMC methods.

## List of Abbreviations

|                        |   |
|------------------------|---|
| AIC                    | Akaike information criterion  |
| BGE–FGM                | Bivariate General Exponential FGM Model                             |
| BGE–I–FGM              | Bivariate General Exponential–General Inverse Exponential FGM Model |
| BGI–E–FGM              | Bivariate General Inverse Exponential–General Exponential FGM Model |
| BGI–FGM                | Bivariate General Inverse Exponential FGM                           |
| BIC                    | Bayesian information criterion                                      |
| cdf                    | Cumulative distribution function                                    |
| FGM                    | Farlie–Gumbel–Morgenstern   |
| $GEF(\alpha, \beta)$   | General Exponential Form  |
| $IEF(\lambda, \omega)$ | General Inverse Exponential Form                                    |
| IFM                    | Inference functions for margins                                     |
| K–S Test               | Kolmogorov–Smirnov Goodness–of–Fit Test                             |
| MCMC                   | Markov Chain Monte Carlo Methods                                    |
| MH                     | Metropolis–Hastings Technique                                       |
| MLE                    | Maximum likelihood estimator  |
| MSE                    | Mean–squared error  |
| pdf                    | Probability density function  |

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