Gauge field theory coupled to gravity

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By

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Abstract

The aim of this thesis is to study gauge theories as a standard tool for describing interactions with gravity.

In chapter 1. we give a general introduction for the unified theories. The theory of supergravity, an elegant formal generalization of Einstein's theory is introduced. The scalar-tensor theory of Brans and Dicke as a theory with varying gravitational constant and the inflationary scenario in the very early universe are discussed. The idea of extended inflation as a modified gravity theory for Brans-Dicke theory is explained. We end this chapter by discussing the induced gravity model which differs from Brans- Dicke theory by the existence of a potential $V(\phi)$ for the scalar field ϕ.

Chapter 2. gives the required mathematical background for the study of gauge theories; such as fiber bundle and connections , since the non-abelian Yang- Mills fields, for example, may be though as the curvature of connection in an SU(2) principle bundle. These gauge fields are invariant under a Lie group of space-time diffeomorphisms. We end this chapter by sec.4.5 in which the definition of characteristic classes as topological invariants of the vector bundle associted to the principle bundle are given.

In chapter 3. we present the coherent state approach to the quantization of the coupled SU(2) Yang-Mills-Einstein system for the conformal Robertson- Walker space-time. It is shown that the coherent state will evolve with time according to the effective potential generated by the non-abelian gauge theory and non trivial topology. It is also shown that there are two physical different phases depending on the gauge coupling value. For small coupling limit $\ell < \ell_c = \frac{1}{\sqrt{2}}$, both non-topological vacuum and topological vacuum are possible. Above the value ℓ_c only the topological vacuum with the fractional Chern-Simon charge 1/2 is possible. These results are published in: Class. and Quant. Gravi. vol. 12 (1995) pp: 3007-3012. see [58]. The vacuum energy (quantum correction) is found to be vanish when $\hbar \rightarrow 0$, and consequently, the Yang-Mills cosmology is identical to the quantum Yang-Mills cosmology.

In chapter 4. the relation between Brans-Dicke theory and the interaction of abelian gauge field with gravity and supergravity is investigated. It is shown that this gauge field produces an external force and has an oscillator in the case of the interaction between gauge field and gravity which is growing by this external force.

It is found also that the scalar factor is inflating and growing for the abelian gauge field produces gravitational field and is inflated by it. These results are given in a talk at the $\overline{XVII}$ International School of Theoretical Physics "Standard Model and Beyond 93" September 23-30, 1993, Szczyrk, Poland., see [96].

Contents

1 General Introduction

For a very long time one of the main objectives of physics has been to find a unifying, common explanation for various, apparently different phenomena. It was Maxwell who unified electricity, magnetism and light. Einstein in his special theory of relativity unified the concepts of space and time, and later he put together geometry and dynamics to create the general theory of relativity (GTR), one of the most impressive achievements of modern physics. A lot of effort is concentrated on attempts to unify the three fundamental interactions in elementary particle physics: electromagnetic, weak and strong one. The first success in this direction is the electro-weak theory of Glashow, Salam and Weinberg (Nobel Lectures. December 8, 1979, published by Pergamon Press [90]) and hopes exist of including also the strong interactions within the frame work of a Grand Unified Theory (GUT). However, the fourth and most universal force-the gravitional one-seems to be ignored by elementary particle physics. It is too weak and characterized by the dimensional coupling constant. Newton's constant $(10^{19}GeV)^{-2}$ (its typical energy scale is given by the Planck mass $M_p \sim 10^{19}(GeV)$, so it should not contribute significantly to processes at present or even foreseeable energies. For instance, in the widely discussed GUT models the unification of strong and electroweak interactions is expected to take place at energies of the order $10^{15}GeV$ [82]; also, near black holes as well as in the early universe (Big Bang) huge gravitational fields are supposed to cause non-trivial quantum effects. This energy is quite close to the Planck mass $M_p \sim 10^{19}GeV$ above which gravitational effects are expected to become important.

The dominant philosophy nowadays is that the four fundamental forces are associated with gauge symmetries. Indeed, the electromagnetic field is the gauge field of local U(1) symmetry, the gauge group SU(2) x U(1) is the basis for the electroweak models, the gluon (strong interaction) fields are the gauge fields of local SU(3) [19], [82]. Finally, the gravitational interaction is believed to be carried by the metric field of general coordinate transformations (also interpreted as the gauge theory of Poincare'group).Moreover, all attempts at constructing GUT's are based on gauge symmetries (e.g SU(5), SO(10), etc..) [19]. One can say that the choice of gauge group is the starting point for building GUT's. In an attempt to describe strong interactions at classical level, C.N.Yang and R.Mills proposed in 1954 [95] that the Lagrangian of the interaction should involve a potential with values in the Lie algebra of the non- Abelian group SU(2) (describing the degrees of freedom of isotopic spin). Moreover, this Lagrangian should be invariant under the group of local internal symmetries, again called gauge trasnformations. Such theories have become known as gauge theories, which we hope to describe all interactions among elementary particles. In fact an SU(3)xSU(2)xU(1) Yang-Mills theory, the so called standard model [14] can provide the basic frame work for unifying electromagnetic, weak and strong interactions.On the other hand, general relativity can be described by a non-Abelian gauge theory. However, trying to unify gravity with the other interactions one has problems with the symmetry involved. The point is that gravity is based on a space-time gauge symmetry whereas the remaining interactions have to do with internal gauge symmetries. Therefore, a new symmetry combining space-time and internal space is required. Again, some years ago such symmetries were believed not to exist ("no-go" theorem) because an internal symmetry can change neither spin nor mass.

Since the discovery of supersymmetry in 1971 by Gol'fand and Likhtman [38] the situation is changed and the way to avoid the strictures of the "no-go-theorem" proved to be the generalization from group of symmetries to graded groups. Supersymmetry is defined as graded extension of the Poincare' algebra and it is the only known way to have a non-trivial unification of space-time and internal symmetries of the scattering matrix in a relativistic particle theory [83], [98].

This leads to the birth of supergravity [35], which is a theory of unification of Einstein's theory of gravitation with matter, particularly of the half-integer spin (fermions). The most straightforward of constructing supergravity theories is to take the fields describing the various particles in the gravitational supermultiplet and try to arrange a supersymmetric interaction among them order by order is the coupling constant k. The first big success was for $N = 1$ pure supergravity (i.e without matter) which is known as the simple supergravity model. Extended supergravity models for $N = 2, 3, 4$ have been also constructed [18] and higher-drivative supergravity and $N = 2$ Yang-Mills theories are investigated [43].

As it should have become apparent, supergravity for the time being is more a general frame work than a concrete physical theory. The extended supergravity theories represent an obvious qualitative success. The gauge principle of extended local supersymmetry provides a way to unify the graviton with the lower spin fields and Einstein gauge invariance with internal symmetry. For instance, within the frame work of simple supergravity coupled to simply supersymmetric matter one can construct models which are almost realistic. Another case which deserves more study is that of $N = 4$ supergravity coupled to $N = 4$ Yang-Mills theory [98]. One may ask the following question: Does supergravity realize Einstein's dream of giving a geometric meaning to all elementary fields? This depends on one's idea of what geometry is. The fact that fields of different spins, including the gravitational field, belong to the same irreducible representation (supermultiplet) of an algebra should satisfy those for whom geometry can be reduced to group theory. But if one is asking for an approach more along the lines of differential geometry, it is important to observe that supergravity can be described in terms of the differential geometry of a supermanifold, superspace [57]. Just like Einstein's theory of gravitation can be obtained by generalizing flat Minkowski space to curved space, similarly supergravity can be described geometrically by studying a suitable superspace, which generalizes the rigid superspace of global supersymmetry.

Thus, we see that supergravity leads to the development of two elegant mathematical techniques, both concerned with an enlargement of the four-dimensional space-time of Einstein theory: the superspace technique, in which the additional coordinates are fermionic, and the more traditional higher dimensional technique of the Kaluza-Klein variety, in which the additional coordinates are bosonic.

Now as an alternative to the standard theory of gravitation is the scalar-tensor theory since all theories unifying the known elementary interactions proposed so far predict the existence of at least one scalar field [34], [24]. It seems inevitable that any scalar tensor theory results in a variable G-theory, a theory which involving the gravitational constant G as a variable with time i.e. $G \sim T^{-2}$ where T is the age of the universe [73],[70],[89]. This led Brans and Dicke [16] to assume that space-time can be represented by Lorentzian manifold and the field equations for their theory are:

$$R_{ab} - \frac{1}{2}Rg_{ab} = \frac{8\pi}{\phi}T_{ab} + \frac{\omega}{\phi^2}(\phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,c}\phi^{,c}) + \frac{1}{\phi}(\phi_{;ab} - g_{ab}\Box\phi)$$

the scalar field ϕ satisfying the wave equation

$$\Box\phi = \phi^{;a}_{;a} = \frac{8\pi}{3+2\omega}T^{a}_{a}$$

where ω is the dimensionless "Dicke coupling constant". For details of the above equations see sec.4.3. Thus in the Lagrangian of Brans and Dicke the coupling constant G of general relativity has been replaced by ϕ^{-1} and instead of attaching it to the matter Lagrangian it is affixed to the geometric scalar R. This effectually means that the equations of motion and the conservation relations are not distributed even though the coupling ϕ is now a field variable. The coupling constant ω is left undetermined which is an obvious weak point of the theory although it gives some scope for manipulation when questions of fit with observational data are taken up. Cosmological solutions of the Brans-Dicke are discussed in sec.4.3 and can be found in [73] chapter 9. More recently, cosmological models with variable gravitational constant G are also found by Arbab [5] and Beesham [11]. A modefied Brans-Dicke gravitational theory is introduced first by Smalley [79],[80] incorprating the assumption of a nonzero divergence of the energy-momentum tensor which depends on the curvature and specifically that it is proportional to the gradient of the scalar curvature. This assumption is established by Rastall [72]. On the other hand Smalley [81] derived a prototype of Rastall's version from a variational principle. However this prototype theory has variable gravitational "constant". This theory is found to be equivalent to be modified general theory of relativity obtained in 1996 by AL-Rawaf and taha [3],[4]. A theory based on derivation of the gravitational field equation without requiring energy- momentum conservation. From this point of view [3] the modefied general theory of relativity may in fact be cast in the form of a variable cosmological model [1].

Again as we have mentioned, all theories unifying the known elementary interactions proposed so far predict the existence of at least one scalar field ϕ. This requeries also to take into account its potential $V(\phi)$. Some types of potential have been extensively used in the literature [2], [31]. Since the last two decade considerable interest appeared in the study of the cosmological concequences of symmetry breaking phase transitions, which occur in grand unified theories (GUT's) with the decrease of temprature at very early stages of the evolution of the universe [55], [46]. These phase transitions typically are strongly first order. The life time of the supercooled symmetric phase $\varphi = 0$ (φ is the Higgs scalar field which breaks the symmetry) in some theories may be extremely larg. In that case the energy-momentum tensor of particles in the phase $\varphi = 0$ almost vanishes in the course of expansion of the universe and the total energy-momentum tensor reduces to the vacuum stress tensor (cosmological term). This leads to an exponential fast expansion of the universe $a \sim e^{Ht}$, where a is the scale factor and H is the Hubble constant at that time [70].

$$H = [(\frac{8\pi}{3M_p^2})V(0)]^{\frac{1}{2}}$$

where $M_p \approx 10^{19}$ GeV is the Plank mass and $V(0)$ is the vacuum energy. Then at some comparatively small temperature T_c the symmetry breaking phase transition takes place, all the vacuum energy $V(0)$ transforms into thermal energy [55],[46], the universe is reheated up to the temperature $T_1 \approx [V(0)]^{\frac{1}{4}}$ and its further evolution proceeds in a standard way [97].

A most detailed discussion of this scenario is contained in a very interesting paper of Guth [40], where it is shown that the existence of a sufficiently long period of exponential expansion (called inflation) in the early universe would provide a natural solution of the horizon and flatness problem in cosmology and of the primordial monopole problem in grand unified theories [70],[54]. This Guth's original inflation scenario, unfortunately leads to some unacceptable consequences, recognized by Guth himself and by other authors who have studied this problem later [41],[21],[8]. The flaw in this "old inflation" model is that the de Sitter expansion never ends. This flaw is sometimes referred as the "graceful exist".

A new type of inflationary scenario based on metric formulations of gravity different from that of Einstein, e.g, a Brans-Dicke theory of gravity is suggested by La and Steinhardt [50] and called it "extended inflation" model. Its key feature is that the effective gravitational constant G varies with time due to non-minimal coupling of a scalar field to the scalar curvature. In this new inflation model, though, graceful exit and successful inflation are attainable through the introductin of this metric theory of gravity of Brans-Dicke for which the action is given by

$$S = \int d^4x\sqrt{-g}(-\frac{R}{16\pi}\Phi + \frac{\omega}{16\pi}g^{\mu\nu}\frac{\partial_\mu\Phi\partial_\nu\Phi}{\Phi} + \mathcal{L}_{matter})$$

where $\Phi = 2\pi\phi^2/\omega$, the kinetic term can be written in the standard way

$$S = \int d^4x\sqrt{-g}(-\frac{R}{8\omega}\phi^2 + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \mathcal{L}_{matter})$$

The Brans-Dicke field Φ couples to gravity and is responsible for the time variation of G. The inflation field σ contributes to $\mathcal{L}_{matter}$ and provides the nearly constant vacuum energy density that derives inflation. ω is a dimensionless parameter of the theory. Brans-Dicke gravity becomes identical to Einstein gravity as ω approaches infinity.

An alternative approach to inflation was suggested by Linde [54] in which a scalar field rolls down a potential from some initial large value and derives the inflation by its potential energy. McDonald [62] studied the case where the scalar field driving the inflation corresponds to a matter scalar which is minimaily coupled to gravity. On the other hand, the extended inflation model failed because the requirement that the production of larg bubbles not disturbe the isotropy of the universe led to the constant $\omega \lesssim 20$, which is at adds with observational costraints $\omega \gtrsim 500$ [48]. However the observational constraint holds in "pure" Brans-Dicke theories and it the BD field has small mass, the constraint will not apply. It is easy to see why: In the limit that $\Phi = const$, Brans-Dicke is equivalent to Einstein gravity. The coefficient of the Φ kinetic term is proportional to ω. As ω becomes large, the kinetic term decouples from the low energy theory and Φ becomes constant. In this way larg ω Brans-Dicke theories are indistinguishable from Einstein. However there is another way to avoid the observational constraint; to give the BD field a mass. If there is some potential keeping Φ anchored at some value (namely $(16\pi G_N)^{-1}$, G_N is the present value of the gravitional constant), then the low-energy limit of Brans-Dicke will again resemble Einstein gravity. Models with potential for Φ are called "induced gravity" models. The key difference between induced gravity and Brans-Dicke theory [2] is the existence of a potential $V(\phi)$ for the scalar field ϕ. Neglecting the potential, the two theories can be mapped into one another by a simple field redefinition, since

$$\mathcal{L}_{BD} = \Phi R + \frac{\omega}{\Phi}(\partial\Phi)^2$$

setting

$$\Phi = \frac{\varepsilon}{2}\phi^2$$

and

$$\omega = (4\varepsilon)^{-1}$$

takes

$$\mathcal{L}_{BD} \rightarrow \mathcal{L}_{IG} = \frac{\varepsilon}{2}\phi^2 R + \frac{1}{2}(\partial\phi)^2$$

The difference arising from the inclusion of a potential is important. Because of $V(\phi)$, at low energies G is strongly anchored at its presently measured value- for low energies the theory is identical to general relativity with the gravitational constant $G = (8\pi < \phi >^2)^{-1}$ Only at very high energies does the theory deviate from general relativity. In sec. 4.5 a study is carried out when the interaction between scalar field and gravity in Brans-Dicke theory is replaced by an interaction between abelian gauge field and supergravity [96].

Very recently Pork and Yoon [69] studies the confermal coupling in induced gravity and they found the induced gravity with confermal couplings requires the confermal invariance in both classical and quantum levels for consistancy.

2 Mathematical Background

2.1 Introduction

It is well known that physical theories use the language of mathematics for their formulation. However, as mathematics and physics have become increasingly specialized over the last several decades, a foremidable language barrier has grown up between the two. It is thus remarkable that several recent developments in theoretical physics have made use of the ideas and results of modern mathematics. In this chapter we give various mathematical ideas, methods and results in differential and Riemannian geometry which are essential for our work in gauge theories.

In section 2.2, we review briefly the basic concepts and definitions in differential and Riemannian geometry. In section 2.3, we discuss in details the idea of fiber bundles since the bundle formalism conveys an intuitive geometric meaning contained in notions like that of a connection on the bundle space or that of curvature. These concepts are, in fact, more clear to understand than those appearing in a Lagrangian theory and are the natural framework in which to discuss gauge theories. Connection in principle fiber bundle is presented in section 2.4. A brief introduction to characteristic classes and homotopy classes is given in section 2.5.

Indeed, there are several references [29] [59] containing the applications of these differential geometric concepts in gauge theories.

2.2 Manifolds and differential forms

In this section, we review briefly the basic concepts of differential geometry, differentiable manifolds and differential forms [47],[76].

Definition 2.2.1

Let M be a connected topological space. A chart (U, Φ) is a pair consisting of an open set $U \subset M$ and a homeomorphism

$$\Phi : U \rightarrow \Phi(U)$$

where $\Phi(U)$ is an open subset of $\Re^n$. M is a topological manifold if M admits a family

$$\{ U_i , \Phi_i \}_{i \in I}$$

of charts such that

$$\{ U_i \}_{i \in I}$$

covers M. This family of charts is said to be an atlas for M.
If (U_i, Φ_i) and (U_j, Φ_j) are two charts and

$$U_{ij} = U_i \bigcap U_j \neq \emptyset$$

then

$$\Phi_{ij} := \Phi_i \circ \Phi_j^{-1} : \Phi_j(U_{ij}) \to \Phi_i(U_{ij})$$

is a homeomorphism. The maps Φ_{ij} are called transition functions. If the Φ_{ij} are C^p-diffeomorphisms (i.e. Φ_{ij} and Φ_{ij}^{-1} are of class C^p, $0 < p < \infty$), then M is called a differential manifold of class C^p. A differentiable manifold is of class C^∞ (i.e. the Φ_{ij} are differentiable). The dimension of M denoted by $\dim(M)$ is the dimension of $\Re^n$.

Definition 2.2.2

The tangent space $T_p(M)$ at the point $p \in M$ is the vector space spanned by the tangents at p to all curves passing through p in the manifold. $T_p(M)$ has an n-dimensional vector space at each point $p \in M$.

Definition 2.2.3

The cotangent space $T_p^*(M)$ of a manifold at $p \in M$ is defined as the dual vector space to the tangent space $T_p(M)$. A dual vector space is defined as follows :
Given an $n-$dimensional vector space V with basis E_i, $i = 1, 2, .., n$; the basis θ^j of the dual space V^* is determined by the inner product

$$< E_i , \theta^j > = \delta_i^j$$

If E_i are the natural basis. i.e.

$$E_i = \frac{\partial}{\partial x_i}$$

for $T_p^j(M)$ we write the basis vectors $T_p^*(M)$ as the differentials $\theta^i = dx^i$. Thus the inner product is given by

$$< \frac{\partial}{\partial x_i} , dx^j > \delta_j^i$$

Now, consider the vector field

$$X = X^i \frac{\partial}{\partial x_i}$$