



شبكة المعلومات الجامعية  
التوثيق الإلكتروني والميكروفيلم

# بسم الله الرحمن الرحيم



**MONA MAGHRABY**



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# شبكة المعلومات الجامعية التوثيق الإلكتروني والميكروفيلم



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# جامعة عين شمس

## التوثيق الإلكتروني والميكروفيلم

### قسم

نقسم بالله العظيم أن المادة التي تم توثيقها وتسجيلها  
علي هذه الأقراص المدمجة قد أعدت دون أية تغيرات



### يجب أن

تحفظ هذه الأقراص المدمجة بعيدا عن الغبار



**MONA MAGHRABY**



# NEW SPECTRAL APPROACHES FOR SOLVING VARIOUS TYPES OF FRACTIONAL DIFFERENTIAL EQUATIONS USING JACOBI POLYNOMIALS

## A Thesis

Submitted in the Fulfillment of the Requirement  
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(Numerical Analysis & Approximation Theory)

## Presented by

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# List of Publications

- **M.M. Alsuyuti**, E.H. Doha, S.S. Ezz-Eldien, I.K. Youssef, Spectral Galerkin schemes for a class of multi-order fractional pantograph equations, J. Comput. Appl. Math., **384** (2021) 113157.
- **M.M. Alsuyuti**, E.H. Doha, S.S. Ezz-Eldien, B.I. Bayoumi, D. Baleanu, Modified Galerkin algorithm for solving multitype fractional differential equations, Math. Methods Appl. Sci., **42**(5) (2019) 1389-1412.
- **M.M. Alsuyuti**, E.H. Doha, B.I. Bayoumi, S.S. Ezz-Eldien, Robust spectral treatment for time-fractional delay partial differential equations, Appl. Math. Comput., (submitted).