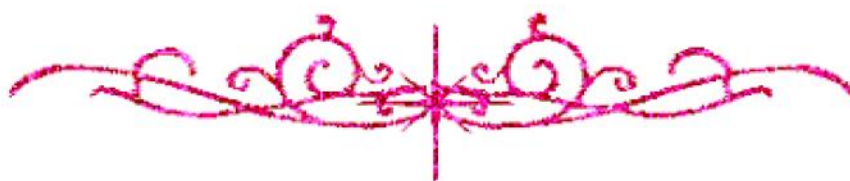


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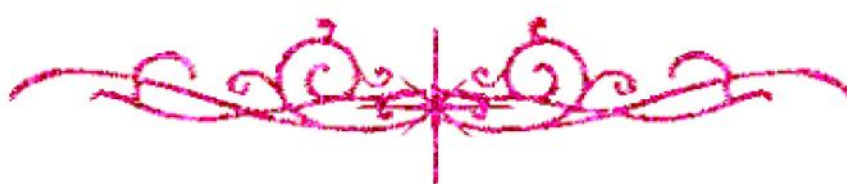
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شبكة المعلومات الجامعية التوثيق الالكتروني والميكرو فيلم



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التوثيق الإلكتروني والميكروفيلم

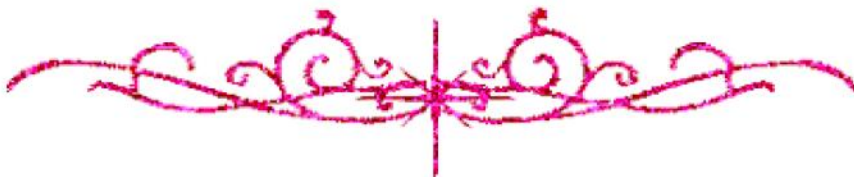
قسم

نقسم بالله العظيم أن المادة التي تم توثيقها وتسجيلها
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تحفظ هذه الأقراص المدمجة بعيدا عن الغبار



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بعض الوثائق الأصلية تالفة



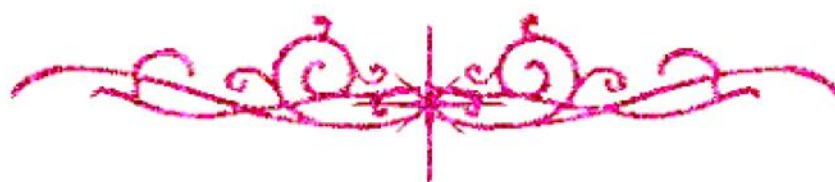
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q -Difference Equations

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Abstract

In this thesis, we study the Cauchy problem of q -difference equations. We distinguish between two cases. The first case is when the initial conditions are defined at $a = 0$. The other is when the initial conditions are defined at $a > 0$. Unlike the case of differential equations, the two cases are different in the case of q -difference equations. We derive existence and uniqueness theorems for both cases based on q -analogues of Picard-Lindelöf method of successive approximations. When $a = 0$, we study the linear q -difference equation of order n . A fundamental set of solutions is derived when the coefficients are all constants. The q -type Wronskian is defined and a q -type Liouville's formula is given. Several illustrative examples are given including q -type Legendre polynomials. The asymptotic formulae for solutions as well as a Sturm-type separation theorem will be given at the end of the thesis.

Notations

$$\mathbb{N} := \{0, 1, 2, \dots\}$$

$$\mathbb{Z}^+ := \{1, 2, 3, \dots\}$$

$$\mathbb{Z} := \mathbb{N} \cup -\mathbb{N}$$

$$\mathbb{R} := (-\infty, \infty)$$

$$\mathbb{R}^+ := (0, \infty)$$

$\mathbb{C}$ is the set of all complex numbers.

f.s. is abbreviation for fundamental set.

Preface

This work is concerned with the basic theory of q -difference equations. It is directed in the way of building up a comprehensive theory of q -difference equations as the available one of the ordinary differential equations. In the following, we give a description of the organization of the thesis.

In Chapter 1, we give a brief account of the main theorems and concepts of ordinary differential equations which we will give their q -analogues in our work like the existence and uniqueness theorem, the range of validity theorem, the Picard-Lindelöf successive approximation method, fundamental set of solutions, the Wronskian, the method of variation of constants (which gives solutions for inhomogeneous equations), Gronwall's inequality, $L^\alpha[a, \infty)$ spaces, $\alpha \geq 0$, $\alpha \geq 1$, the asymptotic behavior of solutions and Sturm separation theorem.

Since a systematic study of the basic theory of q -difference equations has not yet been developed, we have to include some q -calculus in Section 2.1. Sections 2.2 and 2.5 are mainly devoted to a study of the existence and uniqueness theorems of the first order systems (2.2.5) and (2.2.6) below. We give an answer to the following question. What is the initial value problem? More specifically, what are the initial data, which we need, in order to obtain a unique solution of any of the first order systems (2.2.5) and (2.2.6) below in a given interval I which contains a point $a \geq 0$.

If $a = 0$ a set of initial conditions at the point zero is sufficient to obtain a unique solution on I . Therefore, the problem, in this case, is similar to the initial value problem in the theory of ordinary differential equations, see [13].

If $a > 0$ the description of the problem depends on the first order system we study. For the first order system (2.2.5) below, a set of initial conditions is not enough to guarantee the uniqueness of the solutions. In this case, a set of "initial functions" has to be known throughout an initial interval left to a . The problem in this case is analogue to the initial value problem in the theory of delay differential equations, see [19]. As for the first order system (2.2.6) below, a set of "terminal functions" on an interval right to a is needed to assure the uniqueness of the solutions. Therefore, in this case, we call the problem the backward value problem.

The proofs of these existence and uniqueness theorems are based on q -analogues of the *Picard-Lindelöf successive approximation method*.

Section 2.3 includes the existence and uniqueness theorem and the range of validity theorem of the general q -difference equation of order n with initial data prescribed at zero. Section 2.4 contains some q -analogues of the exponential functions and the sine and cosine functions, introduced by Jackson in [25]. Also, we introduce another q -analogue of the sine and cosine functions as solutions of the second-order q -difference equation $-\frac{1}{q}D_{q^{-1}}D_q y(x) = y(x)$, with the initial data $y(0) = 0$, $D_q y(0) = 1$ and $y(0) = 1$, $D_q y(0) = 0$ respectively.

In Chapter 3, we study the linear q -difference equations of order n when the coefficients are defined and continuous in an interval that contains zero. The q -type Wronskian is defined in the n th order case extending the results of Swarttouw-Meijer [36]. A q -Liouville-type formula for the Wronskian is derived and q -type Legendre

polynomials are constructed. A fundamental set of solutions is derived for the n th order linear q -difference equations with constant coefficients. Also, a q -analogue of the method of variation of parameters (constants) to solve inhomogeneous equations is given. In Section 3.5, we study the properties of the solutions of a q -type Sturm Liouville equation in an interval of the form $[a, \infty[$, $a \geq 0$.

In Chapter 4, we aim to investigate the asymptotic behavior of solutions of the q -Sturm Liouville equations. So, a definition for the q -integration on intervals in the form $[x, \infty)$, $x > 0$, is introduced and a definition for the Banach spaces $L_q^\alpha[a, \infty)$, $a \geq 0$ and $\alpha \geq 1$, is developed. Also a derivation of a q -analogue of the Gronwall's inequality is given. In Section 4.4 we give a q -analogue of the Sturm separation theorem. Finally, in Section 4.5, we show that there are continuous functions on closed intervals such that their q -integration can not be handled by Definition 2.1.5 below. Therefore, we suggest an alternative definition for the q -integration for those functions.

Chapter 1

Introduction

In this chapter we briefly introduce the main concepts and theorems of ordinary differential equations as stated in [15], of which we shall give q -analogous.

1.1 System of differential equations

In the following we give a definition of a system of differential equations and its solutions.

Definition 1.1.1. Let r, s , and n_i ($i = 0, 1, \dots, r$) be positive integers and let $N = (n_0 + 1) + \dots + (n_r + 1) - 1$. Let $F_j(x, y_0, y_1, \dots, y_N)$ ($j = 0, 1, \dots, s$) be real-or complex-valued functions, where x is a real variable lying in some interval $I \subseteq \mathbb{R}$ and each y_i is a complex variable lying in some region D_i of the complex plane. If there is a sub-interval J of I and real or complex-valued functions $\phi_i(x)$ ($0 \leq i \leq r$) defined in J such that

- (i) $\phi_i(x)$ has n_i derivatives in J for $0 \leq i \leq r$;
- (ii) $\phi_i^{(m)}(x)$ is in the appropriate D_i for all x in J , $0 \leq m \leq n_i$, and $0 \leq i \leq r$, so that the left-hand side of equation (1.1.1) below is defined;
- (iii) for all x in J and $0 \leq j \leq s$,

$$F_j \left(x, \phi_0(x), \phi_0^{(1)}(x), \dots, \phi_0^{(n_0)}(x), \dots, \phi_r(x), \dots, \phi_r^{(n_r)}(x) \right) = 0, \quad (1.1.1)$$