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بسم اللّه الرحمن الرحيم

مركز الشبكات وتكنولوجيا المعلومات

قسم التوثيق الإلكتروني

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جامعة عين شمس

التوثيق الإلكتروني والميكروفيلم

قسم

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بعض الوثائق الأصلية تالفة وبالرسالة صفحات لم ترد بالأصل

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EFFICIENT AND ACCURATE NUMERICAL METHODS FOR THE BURGERS' AND RELATED PARTIAL DIFFERENTIAL EQUATIONS

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SUMMARY

The main aim of this study is to find a numerical solution to the Burgers equation and the related partial differential equations, such as the modified Burgers equation, and the Korteweg de-Vries -Burgers equation.

In chapter one we introduced the physical meaning of the Burgers equation, and a short note about J. M. Burgers, who discovered this equation, and also the relation to the turbulence and the shock waves, and we established the Burgers equation.

In chapter two, we introduced the analytic solution of Burgers equation through its relation to the Heat equation, the translation property of the Burgers equation, gave some theorems with proofs. We discussed the travelling wave solution to Burgers equation, we proved that the solution is exist and unique for some initial and boundary conditions, and we proved that the energy of the initial boundary value problem associated with the Burgers equation is bounded, which means that the solution is convergent.

In chapter three, we introduced a numerical review about the authors who solved the Burgers equation numerically before this work.

In chapter four, we began our discussion numerically by using the splitting method with two and three levels, and give some graphs to prove that our solution is convergent and also stable. In the last section in this chapter we solved the Burgers equation by using finite difference method (Crank-Nicolson) method and also discussed its stability and convergence according to the L_2 and L_∞ errors norm which remain small.

In chapter five we discussed an important related partial differential equations to the Burgers equation which is the Korteweg de-Vries -Burgers (KdVB) equation, by using the Collocation with quintic spline method. We also discussed the steady state solution to this equation, and its exact solution that appears very near, we also discussed the stability of our method which becomes unconditionally stable. In the last section in this chapter we applied a test problem to discuss the accuracy of our method.

We found that the errors were very small with respect to the coefficients of the dispersive and dissapative terms.

In chapter six, we solved the modified Burgers equation by the same method in the last chapter, and also discussed the stability and convergence of this method, and applied a test problem to investigate the accuracy of our solution.

We drew our conclusion and then wrote the references that we used in this work.

CHAPTER ONE

PHYSICAL MEANING OF THE BURGERS EQUATION

Introduction

In this chapter we give a short information about Burgers, and the physical meaning of his equation known as Burgers' equation.

1-1 Why do we choose the Burgers equation?

Burgers' equation is used frequently in viscosity-dominated systems. It is a famous equation for several reasons, Firstly, it includes non-linearity and dissipation together in the simplest possible way and may be thought of as a non-linear version of the heat equation. Secondly, the analytic solution of the one space variable Burgers equation is available through a transformation known as the Cole-Hopf transformation [1]. Thus Burgers equation provides an important test for many proposed numerical methods dealing with non-linear partial differential equations. Burgers equation is probably one of the simplest non-linear partial differential equations for which it is possible to obtain exact solutions under different initial, and boundary conditions. It behaves as an elliptic, parabolic or hyperbolic partial differential equation. Therefore, Burgers equation has been used widely as a model equation for testing and comparing computational techniques.

Burgers' equation is

$$u_t + uu_x - \nu u_{xx} = 0 \qquad (1.1)$$

Where in a typical application, u is a velocity-like dependent variable and the positive number ν is a viscosity-like parameter which means the inverse of Reynolds number. The subscripts t and x indicate the time and spatial derivatives respectively.

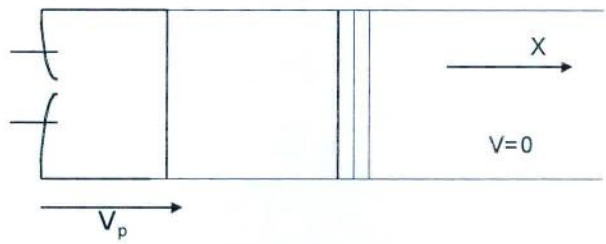
The first solutions (steady) to equation (1.1) were given by Bateman [2]. However, the equation takes its name from the extensive research of Burgers [3], and [4], and [5] in modelling turbulence. The equation is a useful mathematical model for such diverse physical problems as shock waves, traffic flow, acoustic transmission in fogs, etc.; in fact any non-linear wave propagation problem subject to dissipation. The dissipation may arise from viscosity, heat conduction, chemical reaction, etc.

Lighthill [6] formally derived equation (1.1) as a second-order approximation to the one-dimensional unsteady Navier-Stokes equations governing the flow through a propagation shock wave.

An important feature of the equation (1.1) is that it is a prototype equation for the balance between the non-linear convection term, uu_x, and the diffusive term νu_{xx}. The ability to handle this balance efficiently is probably the single most difficult aspect of computing fluid dynamic problems.

1-2 Shock Waves

Figure 1.1 shows us an impulsively started piston moving at a steady velocity, V_P, into a tube containing a compressible fluid, e.g. air, initially at rest. The equation that was governing the one-dimensional unsteady motion of the fluid in the tube will be reduced to the Burgers equation.



Coalescing compression waves produce a Shock Wave

Figure 1-1 Incipient shock wave

The motion of the piston creates compression waves that Propagate to the right at a velocity greater than V_P. The compression waves eventually coalesce, due to the non-linear nature of the convection, to form a single shock wave. Ahead (to the right) of shock wave fluid is undisturbed, $V = 0$. Behind the shock (i.e. between the shock wave and the piston) the fluid moves with the speed of the piston, V_P. As the velocity profile, associated with the coalescence of the compression waves, steppe's due to non-linear convection effect, viscous forces come into play tending to smooth the velocity profile through the shock wave and, by implication, spread the thickness over a finite width. Thus the velocity undergoes a rapid increase through the shock to reach a steady value adjacent to the piston. However, the shock location is a function of time, since the shock travels at a speed greater than V_P. This problem is governed by the continuity equation

$$\rho_t + \rho V_x + V\rho_x = 0 \tag{1.2}$$

Where ρ is the density and V is the velocity. The x-momentum equation can be written

$$V_t + V.V_x + P_x / \rho = \delta.V_{xx} \tag{1.3}$$

Where δ is the diffusivity of sound [6]. It is assumed that entropy changes are small. It is convenient to replace the density by the sound speed, a, *via*

$$a/a_0 = (\rho/\rho_0)^{\frac{1}{2}(\gamma-1)} \tag{1.4}$$

Where γ is the specific heats ratio and subscript 0 refers to the undisturbed (constant) values. Equation (1.2) and (1.3) become

$$V_t + V.V_x + \frac{2}{\gamma-1}.aa_x = \delta V_{xx} \tag{1.5}$$

Where γ is the diffusivity of sound, is a function of undisturbed (to the right of the shock) viscosity, density, specific heat and thermal conductivity of the medium.

Equation (1.4) and (1.5) can be simplified by introducing the Riemann invariant

$$r = \frac{a}{\gamma-1} + \frac{V}{2} \tag{1.6a}$$

$$s = \frac{a}{\gamma-1} - \frac{V}{2} \tag{1.6b}$$

to give:

$$r_t + (a+V)r_x = \frac{1}{2}\delta(r_{xx} - s_{xx}) \tag{1.7}$$

$$s_t - (a-V)s_x = \frac{1}{2}\delta(s_{xx} - r_{xx}) \tag{1.8}$$

For the particular problem under consideration, the propagation of a disturbance into an initially undisturbed region, $s = s_0$ where $s_0 = \dfrac{a_0}{(\gamma-1)}$ from equation (1.6a), and (1.6b).

The problem is governed by equation (1.7). But from equation (1.6),

$$(a+v) = \frac{\gamma+1}{2}r + \frac{1}{2}(\gamma-3)s_0 \tag{1.9}$$

So that equation (1.7) becomes

$$r_t + \left\{\tfrac{1}{2}(\gamma+1)r + \tfrac{1}{2}(\gamma-3)s_0\right\}r_x = \tfrac{\delta}{2}r_{xx} \tag{1.10}$$

As the final step we introduce

$$u = \frac{1}{2}(\gamma+1)(r-r_0) = \frac{1}{2}(\gamma+1)r + \frac{1}{2}(\gamma-3)s_0 - a_0 \tag{1.11}$$

and

$$\xi = x - a_0 t \tag{1.12}$$

to give the Burgers equation

$$u_t + u.u_\xi = \frac{\delta}{2}u_{\xi\xi} \tag{1.13}$$

From equation (1.11), $u \approx a + v - a_0$, i.e., an excess wavelet velocity. The co-ordinate ξ is measured relative to a frame of reference moving with the undisturbed speed of sound a_0.

1-3 Turbulence

In an attempt to formulate a simple mathematical model that would exhibit the essential features demonstrated by turbulence in hydrodynamic flow. Burgers [4], introduced the equation that has now come to be known as the Burgers equation.

In considering turbulent flow in a channel. Burgers introduced two variables U,V where U was a function of time only and was equivalent to a mean flow and V was a function of y (measured across the channel) and time, t, and was equivalent to a fluctuating or turbulent motion. Burgers took as governing equations

$$bU_t = p - VU/b - (\int_0^b V^2 dy)/b \tag{1.14}$$

and

$$V_t + 2V.V_y - V.V_{yy} = VU/b \tag{1.15}$$

In equation (1.14) the term represents the external forces and the last term on the right-hand side is associated with the Reynolds stress. The viscous term VU/b is introduced into both equations as a mechanism by which energy can be transferred from the primary motion to the secondary motion.

1-4 who is J.M.Burgers?

Around 1915 the still young Technological University of Delft realised that fluid mechanics was becoming increasingly important for the technical sciences. Remember, only about ten years earlier the first manned flight of the Orville Wright had taken place and the future potential of flying had started to dawn upon the world. So the university consulted the famous Dutch physicist, H.A. Lorentz. His reaction is said to have been appoint *Jan Burgers*. Indeed an excellent suggestion! No doubt of ***Johannes Martinus Burgers*** was a remarkable man. He studied physics at the University of Leiden from 1914-1918. On 7 November 1918 he obtained his Ph. D. degree from professor, Paul Ehrenfest with a thesis under title: The atomic model of Boher-Rutherford. Even before his formal thesis defence he started to work at the "Technische Hogeschool". Was his 23 years was one of the youngest professors. He quickly familiarised himself with the basic facts of fluid mechanics known at that time. He presented his plans in his inaugural speech:" The hydrodynamic pressure" and almost immediately started to publish his own research results. One of the first subjects in fluid mechanics, which drew his attention, was turbulence. His work in this field culminated in the publication in 1939 of *mathematical examples illustrating relations occurring in the theory of turbulent fluid motion*. In this report, on which he already had started to work since 1933, he presented several simple models for turbulence, known ever since as the *Burgers equation*. It is one of the few non-linear partial equations that allow a full exact solution. Although we know now that the solution of the Burgers equation cannot represent three-dimensional turbulence in flows, it is nevertheless still used to test ideas and concepts in turbulence theory. He was one of the first to apply hot-wire technology to the problem of transition and turbulence in a boundary layer. This work done together with his assistant at that time, B.G. van deer Hegge Zijnen, is still well known. Burgers did not stop at