# بسم الله الرحمن الرحيم

∞∞∞

**تم رفع هذه الرسالة بواسطة / هناء محمد علي**

**بقسم التوثيق الإلكتروني بمركز الشبكات وتكنولوجيا المعلومات دون أدنى مسئولية عن محتوى هذه الرسالة.**

| ملاحظات: |
| --- |
| ........................................................................................................ |
| ........................................................................................................ |
| ........................................................................................................ |
| ........................................................................................................ |
| ........................................................................................ |

Ain Shams University
University College for Women
(Arts, Science, and Education)
Department of Mathematics

# On the Oscillation and Non-oscillation of Solutions of Linear and Nonlinear Differential Equations.

**A THESIS**
**SUBMITTED IN PARTIAL FULFILLMENT OF REQUIREMENTS**
**FOR THE DEGREE OF**
**Ph. D. OF SCIENCE**
**(Pure MATHEMATICS)**

**BY**

**Samah Saad Mahmoud Euat Tallah**

**Department of Mathematics**
**University College for Women**
**Ain Shams University**

**SUPERVISORS**

**Prof. Dr. Mohamed M. A. El-Sheikh**
**Professor of Pure Mathematics**
**Faculty of Science**
**Menoufia University**

**Prof. Dr. Gamal A. F. Ismail**
**Professor of Pure Mathematics**
**University College for Women(Science)**
**Ain Shams University**

**( 2021 )**

Ain Shams University
University College for Women
(Arts, Science, and Education)
Department of Mathematics

Ph. D. Thesis
(Pure MATHEMATICS)

Title of Thesis:

# On the Oscillation and Non-oscillation of Solutions of Linear and Nonlinear Differential Equations.

Thesis supervisors:

**Prof. Dr. Mohamed M. A. El-Sheikh**
Professor of Pure Mathematics
Faculty of Science
(Science)
Menoufia University

**Prof. Dr. Gamal A. F. Ismail**
Professor of Pure Mathematics
University College for Women

Ain Shams University

Ain Shams University
University College for Women
(Arts, Science, and Education)
Department of Mathematics

# COURSES

The student has passed the following courses in partial fulfillment of the requirement for a Ph. D. degree:

1. **Selected Topics (1) Math791. 3h per week**
2. **Linear and Nonlinear Analysis Math710. 3h per week**

Head of Mathematics Department

# Contents

**2-The Oscillation of Second Order Differential Equations With and Without Damping.**



**3. Interval Oscillation of Second Order Forced Differential Equation.**



**4. Philos-Kamenev Oscillation for Impulsive Differential Equations.**

# Abstract

Samah Saad Mahmoud Euat Tallah. On the oscillation and non-oscillation of solutions of linear and nonlinear differential equations. Philosophy degree of science dissertation of Pure Mathematics (Ph. D), University College for Women, Ain Shams University.

The main purpose of thesis is to study; discuss the oscillation and non-oscillation of solutions of linear and non-linear of differential equations. Also, we study different classes of impulsive second order differential equations. Finally, we discuss the oscillation of first order neutral differential equation under impulse effect. Our analysis is based on Riccati transformations and the integral average technique. Applications are provided to illustrate our results obtained in the case of impulse or without impulse effect.

**Keywords:** oscillation, non-oscillation, Riccati transformation, ordinary differential equation, impulsive, neutral differential equation.

# Summary

In recent years the study of the oscillatory behavior of solutions of second-order differential equations has received a lot of attention in the literature due to its importance in different applications in physics, mathematics, engineering and biology, etc. This thesis deals with the oscillatory behavior of different classes of differential equations. Also, it discusses the oscillation of impulsive differential equations due to its importance in dynamical models in the case of the discontinuous movements. This thesis is devoted to establish necessary and sufficient of oscillation and non-oscillation of linear and non-linear differential equations.

This thesis consists of six chapters organized as follows:

**Chapter 1**

In this chapter we outline some basic concepts of oscillation and non-oscillation of differential equations. Basic notations and definitions of second order linear and non-linear differential equations are given. Also, we present some preliminary results of a class of first order neutral differential equations.

**Chapter 2**

In this chapter, we discuss the oscillatory behavior of solutions of general classes of second order forced nonlinear differential equations with and without damping effect. The results are proved by using Riccati transformation, integral average technique and Holder inequality. Also, examples are given to illustrate our results.

**Chapter 3**

In this chapter, we establish the interval oscillatory behavior of solutions of more general classes of second order nonlinear forced differential equations.

The main results will be proved by using different Kamenev auxiliary functions. Also, an example is given.

The main results of chapter (2) and chapter (3) are submitted to **“Mathematica slovaca”**

## Chapter 4

In this chapter, we discuss the oscillation of second order impulsive differential equations using Riccati transformations technique. New oscillation criteria are established, to improve and extend some recent results in the literature. Two illustrative examples are given.

The contents of this chapter have been **published in “Journal of Mathematics and computer science (26) 2022, 172-183”.**

## Chapter 5

In this chapter, we study the oscillation of more general second order impulsive differential equations. New oscillation criteria are established, to improve and extend some previous results.

## Chapter 6

In this chapter we give necessary and sufficient conditions for the oscillatory behavior of a class of first order neutral differential equations with variable coefficients under the impulses effect.

# Chapter 1

## 1.1 Introduction

The topic of differential equations is considered as one of the most important parts of mathematics. The differential equation is such equation which contains derivatives with respect to one or more variables. The importance of differential equations appears from the derivative which represents the rate or the change at certain time and place. Several applications can be represented by differential equations as medical science for modeling the spread of diseases, the electricity movement, in describing the motion of waves and heat conduction. In this thesis, we introduce some new criteria of oscillation and non-oscillation of different types of second order differential equations.

The oscillatory behavior of solutions of linear and non-linear differential equations has received considerable interest in the last three decades due to their applications in physics, mathematics, engineering and biology, etc. Recently, there are many results appeared in the oscillation and non-oscillation of solutions of numerous kinds of differential equations see ([4], [5], [7], [23]).

In this chapter, we outline some basic definitions, notations and preliminary facts. In section 1.2, we introduce some basic lemmas needed to prove our main results. We give some notations for first order impulsive neutral DE and for different classes of second order impulsive differential equations.

**Definition 1.1.1. [5]**

A nontrivial solution of differential equation is called oscillatory if it has an infinite set of zeros; otherwise, it is said to be non-oscillatory. In other words, a non-oscillatory solution must be eventually positive or eventually negative. A differential equation is called oscillatory if all its solutions are oscillatory.

**Examples 1.1.2. [ 5]**

(i) The differential equation $x'' + x = 0$, is oscillatory since it has the solutions $x_1 = cos\, t\,, x_2 = sin\, t$.

(ii) The differential equation $x'' - \frac{1}{t}x' + 4t^2 x(t) = 0$, has the solution $x = sin\, t^2$. It is oscillatory but not periodic.

(iii) The differential equation $x'' + \frac{1}{2}x' - \frac{1}{2}x(t-\pi) = 0, t \geq 0,$ its solution is $x = 1 - sin\, t$ . This solution has an infinite sequence of multiple zeros, which has the oscillatory property.

(iv) The differential equation $x''(t) - x(-t) = 0$, has an oscillatory solution $x_1 = sin\, t$, and a non-oscillatory one $x_2 = e^t + e^{-t}$.

**1.1.3. Riccati equation.[4]**

It is an equation of the form

$$y'(x) = h(x) + p(x)y + g(x)y^2$$

where $h(x), p(x) and\ g(x) \neq 0$.

The non-linear Riccati equation can be converted to a second order **linear** ordinary differential equation.

## 1.2. Some basic inequalities.

The following lemmas involve some inequalities which will be needed in the proof of our results.

**Lemma 1.2.1. [37]**

Let $\gamma \geq 1$ be a ratio of two odd numbers. Then

$$A^{1+\frac{1}{\gamma}} - (A - B)^{1+\frac{1}{\gamma}} \leq \frac{B^{\frac{1}{\gamma}}}{\gamma}[(\gamma + 1)A - B] \tag{1.2.1}$$

$$C^{\frac{\gamma+1}{\gamma}} - \frac{\gamma+1}{\gamma} CD^{\frac{1}{\gamma}} \geq D^{\frac{\gamma+1}{\gamma}}. \tag{1.2.2}$$

**Lemma 1.2.2. (Holder inequality).** [16]

If A and B are nonnegative, then

$$\frac{1}{p}A^p + \frac{1}{q}B^q \geq AB, \text{ for } \frac{1}{p} + \frac{1}{q} = 1.$$

**Lemma 1.2.3.** [30]

Suppose that the functions y(t)and z(t) $\in C(R^+, R^+)$ such that $z(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} z(t) = \infty$. If the function y(t) satisfies

$$\lim_{t\to\infty} inf \int_{z(t)}^{t} y(s)ds > \frac{1}{e},$$

then the inequality $x'(t) + y(t)x(z(t)) \leq 0$ has no eventually positive solution.

**Lemma 1.2.4. (Gronwall's inequality)[4]**

Consider the interval of real numbers $I = [t_0, T)$ . Let

$$v(t) \leq c + \int_{t_0}^{t} q(s)v(s)ds$$

for $t \in I, c \geq 0,$ and $v, q \in C[I, R^+]$. Then

$$v(t) \leq c \exp(\int_{t_0}^{t} q(s)v(s)ds).$$

## 1.3. Previous results for second order non-linear DEs.

The oscillation of second-order differential equations became a very attractive topic for many authors. Kong and Pasic [33] collected the most important obtained results in this topic. In this following section, we establish some of those results.

The first oscillation criteria including integral mean were devoted to the linear unforced differential equation:

$$(r(t)x')' + q(t)x = 0 \qquad (1.3.1)$$

In 1949, Aurel Wintner [62] established the following result.

**Theorem 1.3.1.[62]**

Let $\mathrm{r(t)} = 1$ and $\mathrm{q} \in \mathrm{c}([t_0, \infty), \mathrm{R})$. If

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(u)duds = \infty, \qquad (1.3.2)$$

then Eq. (1.3.1) is oscillatory

The following example shows may show that Wintner' s [62] criterion (1.3.2) is very nice.

**Example 1.3.2.[62]**

Consider the differential Equation:

$$x''(t) + \frac{k}{tlnt} x = 0 \quad , \mathrm{k} > 0, t > 1.$$

Since

$$\int \frac{du}{ulnu} = \ln(lnu), \lim_{u\to\infty}\big(ln(lnu)\big) = \infty \text{ and } \lim_{t\to\infty}\frac{\int_{t_0}^{t} ln(lnu)du}{t} = \infty.$$

Thus the function $q(t) = \frac{k}{tlnt}$ satisfies the criterion (1.3.2) and so Eq. (1.3.1) is oscillatory for any $k > 0$.

**Remark 1.3.3.** One many note that Sturm Comparison principle proves that Eq. (1.3.1) is oscillatory but for $k > \frac{1}{4}$, since for any $\lambda \in \left(\frac{1}{4}, k\right)$,

we have $\frac{k}{tlnt} \geq \frac{\lambda}{t^2}$ , $t > 1$ and by Euler equation: $x''(t) + \frac{\lambda}{t^2}x = 0$ is oscillatory.

In 1968, William Coles [17] introduced the so called weighted average

$$\frac{1}{\int_{t_0}^{t} w(u)du}\int_{t_0}^{t} w(u)q(u)du,$$

where $\int_{t_0}^{t} w(u)du \neq 0$

**Theorem 1.3.4.[17]** Suppose that the assumption of Theorem 1.3.1 be satisfied. Further if there exists a weighted function *w(t)* satisfying

$$\lim_{t\to\infty}\int_{t0}^{t}\frac{w(s)(\int_{t_0}^{s} w(u)du)^k}{\int_{t_0}^{s} w^2(u)du}ds = \infty\,, \tag{1.3.3}$$

and $\lim_{t\to\infty}\frac{1}{\int_{t_0}^{t} w(s)ds}\int_{t_0}^{t} w(s)\int_0^s q(u)du\,ds = \infty,$ (1.3.4)

then the differential Eq. (1.3.1) is oscillatory.

**Remark 1.3.5.** It is easy to show that the differential equation

$$x'' + t^2(sin\, t)x = 0,$$

is oscillatory, but it cannot be obtained by Theorem 1.3.1. By choosing the weight function $w(t) = t$ and the fact that: